Solution to Chapter 5 Problems

Problem 5.1

Let us denote by r_n (b_n) the event of drawing a red (black) ball with number n. Then

$$\begin{aligned}
1.\ E_1 &= \{r_2, r_4, b_2\} \\
2.\ E_2 &= \{r_2, r_3, r_4\} \\
3.\ E_3 &= \{r_1, r_2, b_1, b_2\} \\
4.\ E_4 &= \{r_1, r_2, r_4, b_1, b_2\} \\
5.\ E_5 &= \{r_2, r_4, b_2\} \cup [\{r_2, r_3, r_4\} \cap \{r_1, r_2, b_1, b_2\}] \\
&= \{r_2, r_4, b_2\} \cup \{r_2\} = \{r_2, r_4, b_2\}
\end{aligned}$$

Problem 5.2

Solution:
Since the seven balls equally likely to be drawn, the probability of each event E_i is proportional to its cardinality.

$$P(E_1) = \frac{3}{7}, \quad P(E_2) = \frac{3}{7}, \quad P(E_3) = \frac{4}{7}, \quad P(E_4) = \frac{5}{7}, \quad P(E_5) = \frac{3}{7}$$

Problem 5.3

Solution:
Let us denote by X the event that a car is of brand X, and by R the event that a car needs repair during its first year of purchase. Then
1)

$$\begin{aligned}
P(R) &= P(A, R) + P(B, R) + P(C, R) \\
&= P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C) \\
&= \frac{5}{100}\frac{20}{100} + \frac{10}{100}\frac{30}{100} + \frac{15}{100}\frac{50}{100} \\
&= \frac{11.5}{100}
\end{aligned}$$

2)

$$P(A|R) = \frac{P(A, R)}{P(R)} = \frac{P(R|A)P(A)}{P(R)} = \frac{.05.20}{.115} = .087$$

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Problem 5.4

Solution:
If two events are mutually exclusive (disjoint) then $P(A \cup B) = P(A) \cup P(B)$ which implies that $P(A \cap B) = 0$. If the events are independent then $P(A \cap B) = P(A) \cap P(B)$. Combining these two conditions we obtain that two disjoint events are independent if

$$P(A \cap B) = P(A)P(B) = 0$$

Thus, at least on of the events should be of zero probability.

Problem 5.5

Let us denote by nS the event that n was produced by the source and sent over the channel, and by nC the event that n was observed at the output of the channel. Then
1)

$$\begin{aligned} P(1C) &= P(1C|1S)P(1S) + P(1C|0C)P(0C) \\ &= .8 \cdot .7 + .2 \cdot .3 = .62 \end{aligned}$$

where we have used the fact that $P(1S) = .7$, $P(0C) = .3$, $P(1C|0C) = .2$ and $P(1C|1S) = 1 - .2 = .8$
2)

$$P(1S|1C) = \frac{P(1C, 1S)}{P(1C)} = \frac{P(1C|1S)P(1S)}{P(1C)} = \frac{.8 \cdot .7}{.62} = .9032$$

Problem 5.6

1) X can take four different values. 0, if no head shows up, 1, if only one head shows up in the four flips of the coin, 2, for two heads and 3 if the outcome of each flip is head.
2) X follows the binomial distribution with $n = 3$. Thus

$$P(X = k) = \begin{cases} \begin{pmatrix} 3 \\ k \end{pmatrix} p^k (1-p)^{3-k} & \text{for } 0 \le k \le 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \begin{pmatrix} 3 \\ m \end{pmatrix} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$

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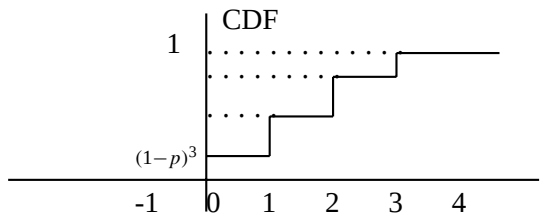

4)

$$P(X > 1) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k} = 3p^2(1-p) + (1-p)^3$$

Problem 5.7

1) The random variables X and Y follow the binomial distribution with $n = 4$ and $p = 1/4$ and $1/2$ respectively. Thus

$$P(X=0) = \binom{4}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4 = \frac{3^4}{2^8} \qquad P(Y=0) = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

$$P(X=1) = \binom{4}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{3^3 4}{2^8} \qquad P(Y=1) = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$P(X=2) = \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{3^3 2}{2^8} \qquad P(Y=2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{2^4}$$

$$P(X=3) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = \frac{3\cdot 4}{2^8} \qquad P(Y=3) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$P(X=4) = \binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 = \frac{1}{2^8} \qquad P(Y=4) = \binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

Since X and Y are independent we have

$$P(X=Y=2) = P(X=2)P(Y=2) = \frac{3^3 2}{2^8}\frac{6}{2^4} = \frac{81}{1024}$$

2)

$$\begin{aligned} P(X=Y) &= P(X=0)P(Y=0) + P(X=1)P(Y=1) + P(X=2)P(Y=2) \\ &\quad + P(X=3)P(Y=3) + P(X=4)P(Y=4) \\ &= \frac{3^4}{2^1 2} + \frac{3^3\cdot 4^2}{2^1 2} + \frac{3^4\cdot 2^2}{2^1 2} + \frac{3\cdot 4^2}{2^1 2} + \frac{1}{2^1 2} = \frac{886}{4096} \end{aligned}$$

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3)

$$\begin{aligned} P(X > Y) &= P(Y=0)\,[P(X=1)+P(X=2)+P(X=3)+P(X=4)]+ \\ &\quad P(Y=1)\,[P(X=2)+P(X=3)+P(X=4)]+ \\ &\quad P(Y=2)\,[P(X=3)+P(X=4)]+ \\ &\quad P(Y=3)\,[P(X=4)] \\ &= \frac{535}{4096} \end{aligned}$$

4) In general $P(X+Y \leq 5) = \sum_{l=0}^{5}\sum_{m=0}^{l} P(X=l-m)P(Y=m)$. However it is easier to find $P(X+Y \leq 5)$ through $P(X+Y \leq 5) = 1-P(X+Y > 5)$ because fewer terms are involved in the calculation of the probability $P(X+Y > 5)$. Note also that $P(X+Y > 5|X=0) = P(X+Y > 5|X=1) = 0$.

$$\begin{aligned} P(X+Y > 5) &= P(X=2)P(Y=4)+P(X=3)[P(Y=3)+P(Y=4)]+ \\ &\quad P(X=4)[P(Y=2)+P(Y=3)+P(Y=4)] \\ &= \frac{125}{4096} \end{aligned}$$

Hence, $P(X+Y \leq 5) = 1 - P(X+Y > 5) = 1 - \frac{125}{4096}$

Problem 5.8

1) Since $\lim_{x\to\infty} F_X(x) = 1$ and $F_X(x) = 1$ for all $x \geq 1$ we obtain $K = 1$.

2) The random variable is of the mixed-type since there is a discontinuity at $x = 1$. $\lim_{\epsilon\to 0} F_X(1-\epsilon) = 1/2$ whereas $\lim_{\epsilon\to 0} F_X(1+\epsilon) = 1$

3)

$$P(\frac{1}{2} < X \leq 1) = F_X(1) - F_X(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

4)

$$P(\frac{1}{2} < X < 1) = F_X(1^-) - F_X(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 1 = 0$$

Problem 5.9

1)

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$$
\begin{aligned}
x < -1 \quad &\Rightarrow \quad F_X(x) = 0 \\
-1 \le x \le 0 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{x} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{x} = \frac{1}{2}x^2 + x + \frac{1}{2} \\
0 \le x \le 1 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{0} (v+1)dv + \int_{0}^{x} (-v+1)dv = -\frac{1}{2}x^2 + x + \frac{1}{2} \\
1 \le x \quad &\Rightarrow \quad F_X(x) = 1
\end{aligned}
$$

2)

$$
p(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{7}{8} = \frac{1}{8}
$$

and

$$
p(X > 0 \big| X < \frac{1}{2}) = \frac{p(X > 0,\ X < \frac{1}{2})}{p(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{1 - p(X > \frac{1}{2})} = \frac{3}{7}
$$

3) We find first the CDF

$$
F_X(x \big| X > \frac{1}{2}) = p(X \le x \big| X > \frac{1}{2}) = \frac{p(X \le x,\ X > \frac{1}{2})}{p(X > \frac{1}{2})}
$$

If $x \le \frac{1}{2}$ then $p(X \le x \big| X > \frac{1}{2}) = 0$ since the events $E_1 = \{X \le \frac{1}{2}\}$ and $E_1 = \{X > \frac{1}{2}\}$ are disjoint. If $x > \frac{1}{2}$ then $p(X \le x \big| X > \frac{1}{2}) = F_X(x) - F_X(\frac{1}{2})$ so that

$$
F_X(x \big| X > \frac{1}{2}) = \frac{F_X(x) - F_X(\frac{1}{2})}{1 - F_X(\frac{1}{2})}
$$

Differentiating this equation with respect to x we obtain

$$
f_X(x \big| X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1 - F_X(\frac{1}{2})} & x > \frac{1}{2} \\ 0 & x \le \frac{1}{2} \end{cases}
$$

4)

$$
\begin{aligned}
E[X \big| X > 1/2] &= \int_{-\infty}^{\infty} x f_X(x|X > 1/2)dx \\
&= \frac{1}{1 - F_X(1/2)} \int_{\frac{1}{2}}^{\infty} x f_X(x)dx \\
&= 8\int_{\frac{1}{2}}^{\infty} x(-x+1)dx = 8(-\frac{1}{3}x^3 + \frac{1}{2}x^2)\Big|_{\frac{1}{2}}^{1} \\
&= \frac{2}{3}
\end{aligned}
$$

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Problem 5.10

In general, if X is a Gaussian RV with mean m and variance σ^2, we have,

$$P(X > \alpha) = Q\left(\frac{\alpha - m}{\sigma}\right)$$

Therefore,

$$P(X > 7) = Q\left(\frac{7-4}{3}\right) = Q(1) = 0.158$$

and using the relation $P(0 < X < 9) = P(X > 0) - P(X > 9)$, we have

$$P(0 < X < 9) = Q\left(\frac{0-4}{3}\right) - Q\left(\frac{9-4}{3}\right) = 1 - Q(1.33) - Q(1.66) \approx 0.858$$

Problem 5.11

1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $P(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned} P(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\ P(X > 4 \times 10^{-4}) &= Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5} \\ P(-2 \times 10^{-4} < X \leq 10^{-4}) &= 1 - Q(1) - Q(2) = .8182 \end{aligned}$$

2)

$$P(X > 10^{-4} | X > 0) = \frac{P(X > 10^{-4},\ X > 0)}{P(X > 0)} = \frac{P(X > 10^{-4})}{P(X > 0)} = \frac{.159}{.5} = .318$$

3) $y = g(x) = xu(x)$. Clearly $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$. If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$. Hence, $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$. $F_Y(y)$ is discontinuous at $y = 0$ and the jump of the discontinuity equals $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2}$$

In summary the PDF $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y)$$

The general expression for finding $f_Y(y)$ can not be used because $g(x)$ is constant for some interval so that there is an uncountable number of solutions for x in this interval.

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4)

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} y \left[f_X(y) u(y) + \frac{1}{2}\delta(y) \right] dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma}{\sqrt{2\pi}} \end{aligned}$$

5) $y = g(x) = |x|$. For a given $y > 0$ there are two solutions to the equation $y = g(x) = |x|$, that is $x_{1,2} = \pm y$. Hence for $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|\text{sgn}(x_1)|} + \frac{f_X(x_2)}{|\text{sgn}(x_2)|} = f_X(y) + f_X(-y) \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \end{aligned}$$

For $y < 0$ there are no solutions to the equation $y = |x|$ and $f_Y(y) = 0$.

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{2\sigma}{\sqrt{2\pi}}$$

Problem 5.12

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \le y \le b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \le b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

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3) In the case of the hard limiter

$$\begin{aligned} P(Y=b) &= P(X<0) = F_X(0) = \frac{1}{2} \\ P(Y=a) &= P(X>0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1-F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$P(Y=y_n) = P(a_n \le X \le a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} \left[F_X(a_{i+1}) - F_X(a_i)\right]\delta(y-x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right)\right]\delta(y-x_i) \end{aligned}$$

Problem 5.13

The equation $x = \tan\phi$ has a unique solution in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, that is

$$\phi_1 = \arctan x$$

Furthermore

$$x'(\phi) = \left(\frac{\sin\phi}{\cos\phi}\right)' = \frac{1}{\cos^2\phi} = 1 + \frac{\sin^2\phi}{\cos^2\phi} = 1 + x^2$$

Thus,

$$f_X(x) = \frac{f_\Phi(\phi_1)}{|x'(\phi_1)|} = \frac{1}{\pi(1+x^2)}$$

We observe that $f_X(x)$ is the Cauchy density. Since $f_X(x)$ is even we immediately get $E[X] = 0$. However, the variance is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - (E[X])^2 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}dx = \infty \end{aligned}$$

Problem 5.14

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1)

$$\begin{aligned} E[Y] &= \int_0^\infty y f_Y(y)dy \geq \int_\alpha^\infty y f_Y(y)dy \\ &\geq \alpha \int_\alpha^\infty y f_Y(y)dy = \alpha P(Y \geq \alpha) \end{aligned}$$

Thus $P(Y \geq \alpha) \leq E[Y]/\alpha$.

2) Clearly $P(|X - E[X]| > \epsilon) = P((X - E[X])^2 > \epsilon^2)$. Thus using the results of the previous question we obtain

$$P(|X - E[X]| > \epsilon) = P((X - E[X])^2 > \epsilon^2) \leq \frac{E[(X - E[X])^2]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Problem 5.15

The characteristic function of the binomial distribution is

$$\begin{aligned} \psi_X(v) &= \sum_{k=0}^{n} e^{jvk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^{jv})^k (1-p)^{n-k} = (pe^{jv} + (1-p))^n \end{aligned}$$

Thus

$$\begin{aligned} E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}(pe^{jv} + (1-p))^n \Big|_{v=0} = \frac{1}{j} n(pe^{jv} + (1-p))^{n-1} pje^{jv} \Big|_{v=0} \\ &= n(p + 1 - p)^{n-1} p = np \\ E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}(pe^{jv} + (1-p))^n \Big|_{v=0} \\ &= (-1)\frac{d}{dv}\left[n(pe^{jv} + (1-p)^{n-1} pje^{jv}\right] \Big|_{v=0} \\ &= \left[n(n-1)(pe^{jv} + (1-p))^{n-2} p^2 e^{2jv} + n(pe^{jv} + (1-p))^{n-1} pe^{jv}\right] \Big|_{v=0} \\ &= n(n-1)(p + 1 - p)p^2 + n(p + 1 - p)p \\ &= n(n-1)p^2 + np \end{aligned}$$

Hence the variance of the binomial distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

Problem 5.16

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The characteristic function of the Poisson distribution is

$$\psi_X(v) = \sum_{k=0}^{\infty} e^{jvk} \frac{\lambda^k}{k!} e^{-k} = \sum_{k=0}^{\infty} \frac{(e^{jv-1}\lambda)^k}{k!}$$

But $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$ so that $\psi_X(v) = e^{\lambda(e^{jv-1})}$. Hence

$$\begin{aligned}
E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}\psi_X(v)\bigg|_{v=0} = \frac{1}{j}e^{\lambda(e^{jv-1})} j\lambda e^{jv}\bigg|_{v=0} = \lambda \\
E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}\psi_X(v)\bigg|_{v=0} = (-1)\frac{d}{dv}\left[\lambda e^{\lambda(e^{jv-1})} e^{jv} j\right]\bigg|_{v=0} \\
&= \left[\lambda^2 e^{\lambda(e^{jv-1})} e^{jv} + \lambda e^{\lambda(e^{jv-1})} e^{jv}\right]\bigg|_{v=0} = \lambda^2 + \lambda
\end{aligned}$$

Hence the variance of the Poisson distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Problem 5.17

For n odd, x^n is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval $[-\infty, \infty]$ is zero, we obtain $E[X^n] = 0$ for n even. Let $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2)dx$ with n even. Then,

$$\begin{aligned}
\frac{d}{dx}I_n &= \int_{-\infty}^{\infty}\left[nx^{n-1}e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2}x^{n+1}e^{-\frac{x^2}{2\sigma^2}}\right]dx = 0 \\
\frac{d^2}{dx^2}I_n &= \int_{-\infty}^{\infty}\left[n(n-1)x^{n-2}e^{-\frac{x^2}{2\sigma^2}} - \frac{2n+1}{\sigma^2}x^n e^{-\frac{x^2}{2\sigma^2}} + \frac{1}{\sigma^4}x^{n+2}e^{-\frac{x^2}{2\sigma^2}}\right]dx \\
&= n(n-1)I_{n-2} - \frac{2n+1}{\sigma^2}I_n + \frac{1}{\sigma^4}I_{n+2} = 0
\end{aligned}$$

Thus,

$$I_{n+2} = \sigma^2(2n+1)I_n - \sigma^4 n(n-1)I_{n-2}$$

with initial conditions $I_0 = \sqrt{2\pi\sigma^2}$, $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We prove now that

$$I_n = 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^n\sqrt{2\pi\sigma^2}$$

The proof is by induction on n. For $n = 2$ it is certainly true since $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We assume that the relation holds for n and we will show that it is true for I_{n+2}. Using the previous recursion we have

$$\begin{aligned}
I_{n+2} &= 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^{n+2}(2n+1)\sqrt{2\pi\sigma^2} \\
&\quad -1 \times 3 \times 5 \times \cdots \times (n-3)(n-1)n\sigma^{n-2}\sigma^4\sqrt{2\pi\sigma^2} \\
&= 1 \times 3 \times 5 \times \cdots \times (n-1)(n+1)\sigma^{n+2}\sqrt{2\pi\sigma^2}
\end{aligned}$$

Clearly $E[X^n] = \frac{1}{\sqrt{2\pi\sigma^2}}I_n$ and

$$E[X^n] = 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^n$$

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Problem 5.18

1) $f_{X,Y}(x, y)$ is a PDF so that its integral over the support region of x, y should be one.

$$\begin{aligned}
\int_0^1 \int_0^1 f_{X,Y}(x, y)dxdy &= K \int_0^1 \int_0^1 (x + y)dxdy \\
&= K \left[\int_0^1 \int_0^1 xdxdy + \int_0^1 \int_0^1 ydxdy \right] \\
&= K \left[\frac{1}{2}x^2 \Big|_0^1 y\Big|_0^1 + \frac{1}{2}y^2 \Big|_0^1 x\Big|_0^1 \right] \\
&= K
\end{aligned}$$

Thus $K = 1$.

2)

$$\begin{aligned}
P(X + Y > 1) &= 1 - P(X + Y \leq 1) \\
&= 1 - \int_0^1 \int_0^{1-x} (x + y)dxdy \\
&= 1 - \int_0^1 x \int_0^{1-x} dydx - \int_0^1 dx \int_0^{1-x} ydy \\
&= 1 - \int_0^1 x(1 - x)dx - \int_0^1 \frac{1}{2}(1 - x)^2 dx \\
&= \frac{2}{3}
\end{aligned}$$

3) By exploiting the symmetry of $f_{X,Y}$ and the fact that it has to integrate to 1, one immediately sees that the answer to this question is 1/2. The "mechanical" solution is:

$$\begin{aligned}
P(X > Y) &= \int_0^1 \int_y^1 (x + y)dxdy \\
&= \int_0^1 \int_y^1 xdxdy + \int_0^1 \int_y^1 ydxdy \\
&= \int_0^1 \frac{1}{2}x^2 \Big|_y^1 dy + \int_0^1 yx \Big|_y^1 dy \\
&= \int_0^1 \frac{1}{2}(1 - y^2)dy + \int_0^1 y(1 - y)dy \\
&= \frac{1}{2}
\end{aligned}$$

4)

$$P(X > Y|X + 2Y > 1) = P(X > Y, X + 2Y > 1)/P(X + 2Y > 1)$$

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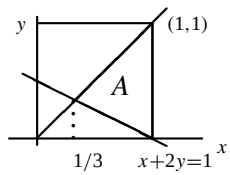
The region over which we integrate in order to find $P(X > Y, X + 2Y > 1)$ is marked with an A in the following figure.



Thus

$$\begin{aligned}
P(X > Y, X + 2Y > 1) &= \int_{\frac{1}{3}}^{1} \int_{\frac{1-x}{2}}^{x} (x + y) dx dy \\
&= \int_{\frac{1}{3}}^{1} \left[x(x - \frac{1-x}{2}) + \frac{1}{2}(x^2 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_{\frac{1}{3}}^{1} \left(\frac{15}{8}x^2 - \frac{1}{4}x - \frac{1}{8} \right) dx \\
&= \frac{49}{108} \\
P(X + 2Y > 1) &= \int_{0}^{1} \int_{\frac{1-x}{2}}^{1} (x + y) dx dy \\
&= \int_{0}^{1} \left[x(1 - \frac{1-x}{2}) + \frac{1}{2}(1 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_{0}^{1} \left(\frac{3}{8}x^2 + \frac{3}{4}x + \frac{3}{8} \right) dx \\
&= \frac{3}{8} \times \frac{1}{3}x^3 \Big|_0^1 + \frac{3}{4} \times \frac{1}{2}x^2 \Big|_0^1 + \frac{3}{8}x \Big|_0^1 \\
&= \frac{7}{8}
\end{aligned}$$

Hence, $P(X > Y | X + 2Y > 1) = (49/108)/(7/8) = 14/27$

5) When $X = Y$ the volume under integration has measure zero and thus

$$P(X = Y) = 0$$

6) Conditioned on the fact that $X = Y$, the new p.d.f of X is

$$f_{X|X=Y}(x) = \frac{f_{X,Y}(x, x)}{\int_0^1 f_{X,Y}(x, x) dx} = 2x.$$

In words, we re-normalize $f_{X,Y}(x, y)$ so that it integrates to 1 on the region characterized by $X = Y$. The result depends only on x. Then $P(X > \frac{1}{2} | X = Y) = \int_{1/2}^{1} f_{X|X=Y}(x) dx = 3/4$.

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7)

$$\begin{aligned} f_X(x) &= \int_0^1 (x+y)dy = x + \int_0^1 ydy = x + \frac{1}{2} \\ f_Y(y) &= \int_0^1 (x+y)dx = y + \int_0^1 xdx = y + \frac{1}{2} \end{aligned}$$

8) $F_X(x|X+2Y>1) = P(X \le x, X+2Y>1)/P(X+2Y>1)$

$$\begin{aligned} P(X \le x, X+2Y>1) &= \int_0^x \int_{\frac{1-v}{2}}^1 (v+y)dvdy \\ &= \int_0^x \left[\frac{3}{8}v^2 + \frac{3}{4}v + \frac{3}{8}\right]dv \\ &= \frac{1}{8}x^3 + \frac{3}{8}x^2 + \frac{3}{8}x \end{aligned}$$

Hence,

$$f_X(x|X+2Y>1) = \frac{\frac{3}{8}x^2 + \frac{6}{8}x + \frac{3}{8}}{P(X+2Y>1)} = \frac{3}{7}x^2 + \frac{6}{7}x + \frac{3}{7}$$

$$\begin{aligned} E[X|X+2Y>1] &= \int_0^1 xf_X(x|X+2Y>1)dx \\ &= \int_0^1 \left(\frac{3}{7}x^3 + \frac{6}{7}x^2 + \frac{3}{7}x\right) \\ &= \frac{3}{7} \times \frac{1}{4}x^4\Big|_0^1 + \frac{6}{7} \times \frac{1}{3}x^3\Big|_0^1 + \frac{3}{7} \times \frac{1}{2}x^2\Big|_0^1 = \frac{17}{28} \end{aligned}$$

Problem 5.19

1)

$$F_Y(y) = P(Y \le y) = P(X_1 \le y \cup X_2 \le y \cup \cdots \cup X_n \le y)$$

Since the previous events are not necessarily disjoint, it is easier to work with the function $1 - [F_Y(y)] = 1 - P(Y \le y)$ in order to take advantage of the independence of X_i's. Clearly

$$\begin{aligned} 1 - P(Y \le y) &= P(Y > y) = P(X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y) \\ &= (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y)) \end{aligned}$$

Differentiating the previous with respect to y we obtain

$$f_Y(y) = f_{X_1}(y)\prod_{i \ne 1}^n (1 - F_{X_i}(y)) + f_{X_2}(y)\prod_{i \ne 2}^n (1 - F_{X_i}(y)) + \cdots + f_{X_n}(y)\prod_{i \ne n}^n (1 - F_{X_i}(y))$$

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2)

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X_1 \le z, X_2 \le z, \cdots, X_n \le z) \\ &= P(X_1 \le z)P(X_2 \le z)\cdots P(X_n \le z) \end{aligned}$$

Differentiating the previous with respect to z we obtain

$$f_Z(z) = f_{X_1}(z)\prod_{i\neq 1}^{n} F_{X_i}(z) + f_{X_2}(z)\prod_{i\neq 2}^{n} F_{X_i}(z) + \cdots + f_{X_n}(z)\prod_{i\neq n}^{n} F_{X_i}(z)$$

Problem 5.20

$$E[X] = \int_0^\infty x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sigma^2}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx$$

However for the Gaussian random variable of zero mean and variance σ^2

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \sigma^2$$

Since the quantity under integration is even, we obtain that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{2}\sigma^2$$

Thus,

$$E[X] = \frac{1}{\sigma^2}\sqrt{2\pi\sigma^2}\frac{1}{2}\sigma^2 = \sigma\sqrt{\frac{\pi}{2}}$$

In order to find $VAR(X)$ we first calculate $E[X^2]$.

$$\begin{aligned} E[X^2] &= \frac{1}{\sigma^2}\int_0^\infty x^3 e^{-\frac{x^2}{2\sigma^2}}dx = -\int_0^\infty x d[e^{-\frac{x^2}{2\sigma^2}}] \\ &= -x^2 e^{-\frac{x^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty 2xe^{-\frac{x^2}{2\sigma^2}}dx \\ &= 0 + 2\sigma^2\int_0^\infty \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = 2\sigma^2 \end{aligned}$$

Thus,

$$VAR(X) = E[X^2] - (E[X])^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = (2 - \frac{\pi}{2})\sigma^2$$

Problem 5.21

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Let $Z = X + Y$. Then,

$$F_Z(z) = P(X + Y \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy$$

Differentiating with respect to z we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{d}{dz} \int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y) \frac{d}{dz}(z - y) dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y) dy \\ &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy \end{aligned}$$

where the last line follows from the independence of X and Y. Thus $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$. With $f_X(x) = \alpha e^{-\alpha x} u(x)$ and $f_Y(y) = \beta e^{-\beta x} u(x)$ we obtain

$$f_Z(z) = \int_0^z \alpha e^{-\alpha v} \beta e^{-\beta(z-v)} dv$$

If $\alpha = \beta$ then

$$f_Z(z) = \int_0^z \alpha^2 e^{-\alpha z} dv = \alpha^2 z e^{-\alpha z} u_{-1}(z)$$

If $\alpha \neq \beta$ then

$$f_Z(z) = \alpha\beta e^{-\beta z} \int_0^z e^{(\beta-\alpha)v} dv = \frac{\alpha\beta}{\beta - \alpha} \left[e^{-\alpha z} - e^{-\beta z} \right] u_{-1}(z)$$

Problem 5.22

1) $f_{X,Y}(x, y)$ is a PDF, hence its integral over the supporting region of x, and y is 1.

$$\begin{aligned} \int_0^{\infty} \int_y^{\infty} f_{X,Y}(x, y) dx dy &= \int_0^{\infty} \int_y^{\infty} K e^{-x-y} dx dy \\ &= K \int_0^{\infty} e^{-y} \int_y^{\infty} e^{-x} dx dy \\ &= K \int_0^{\infty} e^{-2y} dy = K(-\frac{1}{2}) e^{-2y} \Big|_0^{\infty} = K \frac{1}{2} \end{aligned}$$

Thus K should be equal to 2.

2)

$$\begin{aligned} f_X(x) &= \int_0^x 2e^{-x-y} dy = 2e^{-x}(-e^{-y}) \Big|_0^x = 2e^{-x}(1 - e^{-x}) \\ f_Y(y) &= \int_y^{\infty} 2e^{-x-y} dy = 2e^{-y}(-e^{-x}) \Big|_y^{\infty} = 2e^{-2y} \end{aligned}$$

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3)

$$\begin{aligned} f_X(x)f_Y(y) &= 2e^{-x}(1-e^{-x})2e^{-2y} = 2e^{-x-y}2e^{-y}(1-e^{-x}) \\ &\neq 2e^{-x-y} = f_{X,Y}(x,y) \end{aligned}$$

Thus X and Y are not independent.

4) If $x < y$ then $f_{X|Y}(x|y) = 0$. If $x \geq y$, then with $u = x - y \geq 0$ we obtain

$$f_U(u) = f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y} = e^{-u}$$

5)

$$\begin{aligned} E[X|Y=y] &= \int_y^\infty xe^{-x+y}dx = e^y \int_y^\infty xe^{-x}dx \\ &= e^y\left[-xe^{-x}\Big|_y^\infty + \int_y^\infty e^{-x}dx\right] \\ &= e^y(ye^{-y}+e^{-y}) = y+1 \end{aligned}$$

6) In this part of the problem we will use extensively the following definite integral

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx = \frac{1}{\mu^\nu}(\nu-1)!$$

$$\begin{aligned} E[XY] &= \int_0^\infty\int_y^\infty xy2e^{-x-y}dxdy = \int_0^\infty 2ye^{-y}\int_y^\infty xe^{-x}dxdy \\ &= \int_0^\infty 2ye^{-y}(ye^{-y}+e^{-y})dy = 2\int_0^\infty y^2e^{-2y}dy + 2\int_0^\infty ye^{-2y}dy \\ &= 2\frac{1}{2^3}2! + 2\frac{1}{2^2}1! = 1 \end{aligned}$$

$$\begin{aligned} E[X] &= 2\int_0^\infty xe^{-x}(1-e^{-x})dx = 2\int_0^\infty xe^{-x}dx - 2\int_0^\infty xe^{-2x}dx \\ &= 2 - 2\frac{1}{2^2} = \frac{3}{2} \\ E[Y] &= 2\int_0^\infty ye^{-2y}dy = 2\frac{1}{2^2} = \frac{1}{2} \\ E[X^2] &= 2\int_0^\infty x^2e^{-x}(1-e^{-x})dx = 2\int_0^\infty x^2e^{-x}dx - 2\int_0^\infty x^2e^{-2x}dx \\ &= 2\cdot 2! - 2\frac{1}{2^3}2! = \frac{7}{2} \\ E[Y^2] &= 2\int_0^\infty y^2e^{-2y}dy = 2\frac{1}{2^3}2! = \frac{1}{2} \end{aligned}$$

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Hence,

$$COV(X,Y) = E[XY] - E[X]E[Y] = 1 - \frac{3}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

and

$$\rho_{X,Y} = \frac{COV(X,Y)}{(E[X^2]-(E[X])^2)^{1/2}(E[Y^2]-(E[Y])^2)^{1/2}} = \frac{1}{\sqrt{5}}$$

Problem 5.23

$$\begin{aligned}
E[X] &= \frac{1}{\pi}\int_0^{\pi}\cos\theta d\theta = \frac{1}{\pi}\sin\theta\Big|_0^{\pi} = 0\\
E[Y] &= \frac{1}{\pi}\int_0^{\pi}\sin\theta d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^{\pi} = \frac{2}{\pi}\\
E[XY] &= \int_0^{\pi}\cos\theta\ \sin\theta\frac{1}{\pi}d\theta\\
&= \frac{1}{2\pi}\int_0^{\pi}\sin 2\theta d\theta = \frac{1}{4\pi}\int_0^{2\pi}\sin x dx = 0\\
COV(X,Y) &= E[XY] - E[X]E[Y] = 0
\end{aligned}$$

Thus the random variables X and Y are uncorrelated. However they are not independent since $X^2+Y^2=1$. To see this consider the probability $p(|X|<1/2, Y\geq 1/2)$. Clearly $p(|X|<1/2)p(Y\geq 1/2)$ is different than zero whereas $p(|X|<1/2, Y\geq 1/2)=0$. This is because $|X|<1/2$ implies that $\pi/3<\theta<5\pi/3$ and for these values of θ, $Y=\sin\theta>\sqrt{3}/2>1/2$.

Problem 5.24

1) Clearly $X>r, Y>r$ implies that $X^2>r^2, Y^2>r^2$ so that $X^2+Y^2>2r^2$ or $\sqrt{X^2+Y^2}>\sqrt{2}r$. Thus the event $E_1(r)=\{X>r, Y>r\}$ is a subset of the event $E_2(r)=\{\sqrt{X^2+Y^2}>\sqrt{2}r\big|X,Y>0\}$ and $P(E_1(r))\leq P(E_2(r))$.

2) Since X and Y are independent

$$P(E_1(r)) = P(X>r, Y>r) = P(X>r)P(Y>r) = Q^2(r)$$

3) Using the rectangular to polar transformation $V=\sqrt{X^2+Y^2}$, $\Theta=\arctan\frac{Y}{X}$ it is proved (see text Eq. 4.1.22) that

$$f_{V,\Theta}(v,\theta) = \frac{v}{2\pi\sigma^2}e^{-\frac{v^2}{2\sigma^2}}$$

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Hence, with $\sigma^2 = 1$ we obtain

$$\begin{aligned} P(\sqrt{X^2+Y^2} > \sqrt{2}r \big| X, Y > 0) &= \int_{\sqrt{2}r}^{\infty} \int_0^{\frac{\pi}{2}} \frac{v}{2\pi} e^{-\frac{v^2}{2}} dv d\theta \\ &= \frac{1}{4} \int_{\sqrt{2}r}^{\infty} v e^{-\frac{v^2}{2}} dv = \frac{1}{4}(-e^{-\frac{v^2}{2}})\Big|_{\sqrt{2}r}^{\infty} \\ &= \frac{1}{4} e^{-r^2} \end{aligned}$$

Combining the results of part 1), 2) and 3) we obtain

$$Q^2(r) \leq \frac{1}{4} e^{-r^2} \quad \text{or} \quad Q(r) \leq \frac{1}{2} e^{-\frac{r^2}{2}}$$

Problem 5.25

The following is a program written in Fortran to compute the Q function

```
      REAL*8    x,t,a,q,pi,p,b1,b2,b3,b4,b5
      PARAMETER        (p=.2316419d+00, b1=.31981530d+00,
     +     b2=-.356563782d+00, b3=1.781477937d+00,
     +     b4=-1.821255978d+00, b5=1.330274429d+00)
C-
            pi=4.*atan(1.)
C-INPUT
      PRINT*,   'Enter -x-'
      READ*,    x
C-
            t=1./(1.+p*x)
            a=b1*t + b2*t**2. + b3*t**3. + b4*t**4. + b5*t**5.
            q=(exp(-x**2./2.)/sqrt(2.*pi))*a
C-OUTPUT
      PRINT*,   q
C-
      STOP
      END
```

The results of this approximation along with the actual values of $Q(x)$ (taken from text Table 4.1) are tabulated in the following table. As it is observed a very good approximation is achieved.

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x	$Q(x)$	Approximation
1.	1.59×10^{-1}	1.587×10^{-1}
1.5	6.68×10^{-2}	6.685×10^{-2}
2.	2.28×10^{-2}	2.276×10^{-2}
2.5	6.21×10^{-3}	6.214×10^{-3}
3.	1.35×10^{-3}	1.351×10^{-3}
3.5	2.33×10^{-4}	2.328×10^{-4}
4.	3.17×10^{-5}	3.171×10^{-5}
4.5	3.40×10^{-6}	3.404×10^{-6}
5.	2.87×10^{-7}	2.874×10^{-7}

Problem 5.26

The joint distribution of X and Y is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2}\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right\}$$

The linear transformations $Z = X + Y$ and $W = 2X - Y$ are written in matrix notation as

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$$

Thus,

$$f_{Z,W}(z, w) = \frac{1}{2\pi \det(M)^{1/2}} \exp\left\{-\frac{1}{2}\begin{pmatrix} Z & W \end{pmatrix} M^{-1}\begin{pmatrix} Z \\ W \end{pmatrix}\right\}$$

where

$$M = A\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} A^t = \begin{pmatrix} 2\sigma^2 & \sigma^2 \\ \sigma^2 & 5\sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma_Z^2 & \rho_{Z,W}\sigma_Z\sigma_W \\ \rho_{Z,W}\sigma_Z\sigma_W & \sigma_W^2 \end{pmatrix}$$

From the last equality we identify $\sigma_Z^2 = 2\sigma^2$, $\sigma_W^2 = 5\sigma^2$ and $\rho_{Z,W} = 1/\sqrt{10}$

Problem 5.27

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{\sqrt{2\pi}\sigma_Y}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \exp[-A]$$

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where

$$\begin{aligned}
A &= \frac{(x-m_X)^2}{2(1-\rho_{X,Y}^2)\sigma_X^2} + \frac{(y-m_Y)^2}{2(1-\rho_{X,Y}^2)\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{2(1-\rho_{X,Y}^2)\sigma_X\sigma_Y} - \frac{(y-m_Y)^2}{2\sigma_Y^2} \\
&= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left((x-m_X)^2 + \frac{(y-m_Y)^2\sigma_X^2\rho_{X,Y}^2}{\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)\sigma_X}{\sigma_Y}\right) \\
&= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2
\end{aligned}$$

Thus

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho_{X,Y}^2}}\exp\left\{-\frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2\right\}$$

which is a Gaussian PDF with mean $m_X + (y-m_Y)\rho\sigma_X/\sigma_Y$ and variance $(1-\rho_{X,Y}^2)\sigma_X^2$. If $\rho = 0$ then $f_{X|Y}(x|y) = f_X(x)$ which implies that Y does not provide any information about X or X, Y are independent. If $\rho = \pm 1$ then the variance of $f_{X|Y}(x|y)$ is zero which means that $X|Y$ is deterministic. This is to be expected since $\rho = \pm 1$ implies a linear relation $X = AY + b$ so that knowledge of Y provides all the information about X.

Problem 5.28

1) Z and W are linear combinations of jointly Gaussian RV's, therefore they are jointly Gaussian too.

2) Since Z and W are jointly Gaussian with zero-mean, they are independent if they are uncorrelated. This implies that they are independent if $E[ZW] = 0$. But $E[ZW] = E[XY](\cos^2\theta - \sin^2\theta)$ where we have used the fact that since X and Y are zero-mean and have the same variance we have $E[X^2] = E[Y^2]$, and therefore, $(E(Y^2) - E(X^2)) \sim \theta\cos\theta = 0$. From above, in order for Z and W to be independent we must have

$$\cos^2\theta - \sin^2\theta = 0 \Longrightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k \in \mathcal{Z}$$

Note also that if X and Y are independent, then $E[XY] = 0$ and any rotation will produce independent random variables again.

Problem 5.29

1) $f_{X,Y}(x, y)$ is a PDF and its integral over the supporting region of x and y should be one.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy \\
&= \int_{-\infty}^{0}\int_{-\infty}^{0}\frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy + \int_{0}^{\infty}\int_{0}^{\infty}\frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{K}{\pi}\int_{-\infty}^{0}e^{-\frac{x^2}{2}}dx\int_{-\infty}^{0}e^{-\frac{y^2}{2}}dx + \frac{K}{\pi}\int_{0}^{\infty}e^{-\frac{x^2}{2}}dx\int_{0}^{\infty}e^{-\frac{y^2}{2}}dx \\
&= \frac{K}{\pi}\left[2(\frac{1}{2}\sqrt{2\pi})^2\right] = K
\end{aligned}$$

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Thus $K = 1$

2) If $x < 0$ then

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{0} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{0} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

If $x > 0$ then

$$\begin{aligned} f_X(x) &= \int_{0}^{\infty} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{0}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

Thus for every x, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which implies that $f_X(x)$ is a zero-mean Gaussian random variable with variance 1. Since $f_{X,Y}(x, y)$ is symmetric to its arguments and the same is true for the region of integration we conclude that $f_Y(y)$ is a zero-mean Gaussian random variable of variance 1.

3) $f_{X,Y}(x, y)$ has not the same form as a binormal distribution. For $xy < 0$, $f_{X,Y}(x, y) = 0$ but a binormal distribution is strictly positive for every x, y.

4) The random variables X and Y are not independent for if $xy < 0$ then $f_X(x) f_Y(y) \neq 0$ whereas $f_{X,Y}(x, y) = 0$.

5)

$$\begin{aligned} E[XY] &= \frac{1}{\pi} \int_{-\infty}^{0} \int_{-\infty}^{0} XY e^{-\frac{x^2+y^2}{2}} dx dy + \frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy \\ &= \frac{1}{\pi} \int_{-\infty}^{0} X e^{-\frac{x^2}{2}} dx \int_{-\infty}^{0} Y e^{-\frac{y^2}{2}} dy + \frac{1}{\pi} \int_{0}^{\infty} X e^{-\frac{x^2}{2}} dx \int_{0}^{\infty} Y e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi}(-1)(-1) + \frac{1}{\pi} = \frac{2}{\pi} \end{aligned}$$

Thus the random variables X and Y are correlated since $E[XY] \neq 0$ and $E[X] = E[Y] = 0$, so that $E[XY] - E[X]E[Y] \neq 0$.

6) In general $f_{X|Y}(x, y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. If $y > 0$, then

$$f_{X|Y}(x, y) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x \geq 0 \end{cases}$$

If $y \leq 0$, then

$$f_{X|Y}(x, y) = \begin{cases} 0 & x > 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x < 0 \end{cases}$$

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Thus

$$f_{X|Y}(x, y) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} u(xy)$$

which is not a Gaussian distribution.

Problem 5.30

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{(x-m)^2 + y^2}{2\sigma^2}\right\}$$

With the transformation

$$V = \sqrt{X^2 + Y^2}, \qquad \Theta = \arctan\frac{Y}{X}$$

we obtain

$$\begin{aligned} f_{V,\Theta}(v, \theta) &= v f_{X,Y}(v\cos\theta, v\sin\theta) \\ &= \frac{v}{2\pi\sigma^2} \exp\left\{-\frac{(v\cos\theta - m)^2 + v^2\sin\theta}{2\sigma^2}\right\} \\ &= \frac{v}{2\pi\sigma^2} \exp\left\{-\frac{v^2 + m^2 - 2mv\cos\theta}{2\sigma^2}\right\} \end{aligned}$$

To obtain the marginal probability density function for the magnitude, we integrate over θ so that

$$\begin{aligned} f_V(v) &= \int_0^{2\pi} \frac{v}{2\pi\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\ &= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\ &= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} I_0(\frac{mv}{\sigma^2}) \end{aligned}$$

where

$$I_0(\frac{mv}{\sigma^2}) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta$$

With $m = 0$ we obtain

$$f_V(v) = \begin{cases} \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} & v > 0 \\ 0 & v \le 0 \end{cases}$$

which is the Rayleigh distribution.

Problem 5.31

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1) Let X_i be a random variable taking the values 1, 0, with probability $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Then, $m_{X_i} = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0 = \frac{1}{4}$. The weak law of large numbers states that the random variable $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ has mean which converges to m_{X_i} with probability one. Using Chebychev's inequality (see Problem 4.13) we have $P(|Y - m_{X_i}| \geq \epsilon) \leq \frac{\sigma_Y^2}{\epsilon^2}$ for every $\epsilon > 0$. Hence, with $n = 2000$, $Z = \sum_{i=1}^{2000} X_i$, $m_{X_i} = \frac{1}{4}$ we obtain

$$P(|Z - 500| \geq 2000\epsilon) \leq \frac{\sigma_Y^2}{\epsilon^2} \Rightarrow P(500 - 2000\epsilon \leq Z \leq 500 + 2000\epsilon) \geq 1 - \frac{\sigma_Y^2}{\epsilon^2}$$

The variance σ_Y^2 of $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ is $\frac{1}{n}\sigma_{X_i}^2$, where $\sigma_{X_i}^2 = p(1-p) = \frac{3}{16}$ (see Problem 4.13). Thus, with $\epsilon = 0.001$ we obtain

$$P(480 \leq Z \leq 520) \geq 1 - \frac{3/16}{2 \times 10^{-1}} = .063$$

2) Using the C.L.T. the CDF of the random variable $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to the CDF of the random variable $N(m_{X_i}, \frac{\sigma}{\sqrt{n}})$. Hence

$$P = p\left(\frac{480}{n} \leq Y \leq \frac{520}{n}\right) = Q\left(\frac{\frac{480}{n} - m_{X_i}}{\sigma}\right) - Q\left(\frac{\frac{520}{n} - m_{X_i}}{\sigma}\right)$$

With $n = 2000$, $m_{X_i} = \frac{1}{4}$, $\sigma^2 = \frac{p(1-p)}{n}$ we obtain

$$\begin{aligned} P &= Q\left(\frac{480 - 500}{\sqrt{2000p(1-p)}}\right) - Q\left(\frac{520 - 500}{\sqrt{2000p(1-p)}}\right) \\ &= 1 - 2Q\left(\frac{20}{\sqrt{375}}\right) = .682 \end{aligned}$$

Problem 5.32

The random variable $X(t_0)$ is uniformly distributed over $[-1 \;\; 1]$. Hence,

$$m_X(t_0) = E[X(t_0)] = E[X] = 0$$

As it is observed the mean $m_X(t_0)$ is independent of the time instant t_0.

Problem 5.33

$$m_X(t) = E[A + Bt] = E[A] + E[B]t = 0$$

where the last equality follows from the fact that A, B are uniformly distributed over $[-1 \quad 1]$ so that $E[A] = E[B] = 0$.

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(A + Bt_1)(A + Bt_2)] \\ &= E[A^2] + E[AB]t_2 + E[BA]t_1 + E[B^2]t_1t_2 \end{aligned}$$

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The random variables A, B are independent so that $E[AB] = E[A]E[B] = 0$. Furthermore

$$E[A^2] = E[B^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{6} x^3 \Big|_{-1}^{1} = \frac{1}{3}$$

Thus

$$R_X(t_1, t_2) = \frac{1}{3} + \frac{1}{3} t_1 t_2$$

Problem 5.34

Since $X(t) = X$ with the random variable uniformly distributed over $[-1 \;\; 1]$ we obtain

$$f_{X(t_1), X(t_2), \cdots, X(t_n)}(x_1, x_2, \ldots, x_n) = f_{X, X, \cdots, X}(x_1, x_2, \ldots, x_n)$$

for all $t_1, \ldots, t_n$ and n. Hence, the statistical properties of the process are time independent and by definition we have a stationary process.

Problem 5.35

1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin ($\tau = 0$). We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$\mathcal{S}(f) = 1.2\text{sinc}^2(f) - \text{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $S(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

Problem 5.36

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The random variable ω_i takes the values $\{1, 2, \ldots, 6\}$ with probability $\frac{1}{6}$. Thus

$$\begin{aligned}
E_X &= E\left[\int_{-\infty}^{\infty} X^2(t)dt\right] \\
&= E\left[\int_{-\infty}^{\infty} \omega_i^2 e^{-2t} u_{-1}^2(t)dt\right] = E\left[\int_0^{\infty} \omega_i^2 e^{-2t} dt\right] \\
&= \int_0^{\infty} E[\omega_i^2] e^{-2t} dt = \int_0^{\infty} \frac{1}{6} \sum_{i=1}^{6} i^2 e^{-2t} dt \\
&= \frac{91}{6} \int_0^{\infty} e^{-2t} dt = \frac{91}{6}(-\frac{1}{2}e^{-2t})\Big|_0^{\infty} \\
&= \frac{91}{12}
\end{aligned}$$

Thus the process is an energy-type process. However, this process is not stationary for

$$m_X(t) = E[X(t) = E[\omega_i]e^{-t}u_{-1}(t) = \frac{21}{6}e^{-t}u_{-1}(t)$$

is not constant.

Problem 5.37

1) We find first the probability of an even number of transitions in the interval $(0, \tau]$.

$$\begin{aligned}
p_N(n = even) &= p_N(0) + p_N(2) + p_N(4) + \cdots \\
&= \frac{1}{1+\alpha\tau} \sum_{l=0}^{\infty} \left(\frac{\alpha\tau}{1+\alpha\tau}\right)^2 \\
&= \frac{1}{1+\alpha\tau} \frac{1}{1 - \frac{(\alpha\tau)^2}{(1+\alpha\tau)^2}} \\
&= \frac{1+\alpha\tau}{1+2\alpha\tau}
\end{aligned}$$

The probability $p_N(n = odd)$ is simply $1 - p_N(n = even) = \frac{\alpha\tau}{1+2\alpha\tau}$. The random process $Z(t)$ takes the value of 1 (at time instant t) if an even number of transitions occurred given that $Z(0) = 1$, or if an odd number of transitions occurred given that $Z(0) = 0$. Thus,

$$\begin{aligned}
m_Z(t) &= E[Z(t)] = 1 \cdot p(Z(t) = 1) + 0 \cdot p(Z(t) = 0) \\
&= p(Z(t) = 1|Z(0) = 1)p(Z(0) = 1) + p(Z(t) = 1|Z(0) = 0)p(Z(0) = 0) \\
&= p_N(n = even)\frac{1}{2} + p_N(n = odd)\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

2) To determine $R_Z(t_1, t_2)$ note that $Z(t + \tau) = 1$ if $Z(t) = 1$ and an even number of transitions occurred in the interval $(t, t + \tau]$, or if $Z(t) = 0$ and an odd number of transitions have taken place in $(t, t + \tau]$ (we are

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assuming $\tau > 0$). Hence,

$$\begin{aligned}
R_Z(t+\tau,t) &= E[Z(t+\tau)Z(t)] \\
&= 1\cdot p(Z(t+\tau)=1, Z(t)=1) + 0\cdot p(Z(t+\tau)=1, Z(t)=0) \\
&\quad +0\cdot p(Z(t+\tau)=0, Z(t)=1) + 0\cdot p(Z(t+\tau)=0, Z(t)=0) \\
&= p(Z(t+\tau)=1, Z(t)=1) = p(Z(t+\tau)=1|Z(t)=1)p(Z(t)=1) \\
&= \frac{1}{2}\frac{1+\alpha\tau}{1+2\alpha\tau}
\end{aligned}$$

As it is observed $R_Z(t+\tau,t)$ depends only on τ and thus the process is stationary. The above is for $\tau > 0$, in general we have

$$R_Z(\tau) = \frac{1+\alpha|\tau|}{2(1+2\alpha|\tau|)}$$

Since the process is WSS its PSD is the Fourier transform of its autocorrelation function, finding the Fourier transform of the autocorrelation function is not an easy task. We can use integral tables to show that

$$\begin{aligned}
\mathcal{S}_Z(f) = &\frac{1}{2}\delta(f) + \frac{1}{4\alpha}\mathrm{sgn}(f)\left[\sin\left(\frac{\pi f}{\alpha}\right) - \cos\left(\frac{\pi f}{\alpha}\right)\right] + \frac{\pi}{4\alpha}\cos\left(\frac{\pi f}{\alpha}\right) \\
&- \frac{1}{2\alpha}\sin\left(\frac{\pi f}{\alpha}\right)\mathrm{Si}\left(\frac{\pi f}{\alpha}\right) - \frac{1}{2\alpha}\cos\left(\frac{\pi f}{\alpha}\right)\mathrm{Ci}\left(\frac{\pi f}{\alpha}\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Si}(x) &= \int_0^x \frac{\sin(t)}{t}\,dt \\
\mathrm{Ci}(x) &= \gamma + \ln(x) + \int_0^x \frac{\cos(t)-1}{t}\,dt
\end{aligned}$$

Finding the power content of the process is much easier and is done by substituting $\tau = 0$ in the autocorrelation function resulting in $P_Z = R_Z(0) = \frac{1}{2}$.

3) Since the process is stationary

$$P_Z = R_Z(0) = \frac{1}{2}$$

Problem 5.38

1)

$$\begin{aligned}
m_X(t) &= E[X(t)] = E[X\cos(2\pi f_0 t)] + E[Y\sin(2\pi f_0 t)] \\
&= E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) \\
&= 0
\end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

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2)

$$\begin{aligned}
R_X(t+\tau,t) &= E[(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau)))\\
&\quad (X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))]\\
&= E[X^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] +\\
&\quad E[XY\cos(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] +\\
&\quad E[YX\sin(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] +\\
&\quad E[Y^2\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)]\\
&= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau)] +\\
&\quad \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))]\\
&= \sigma^2\cos(2\pi f_0\tau)
\end{aligned}$$

where we have used the fact that $E[XY] = 0$. Thus the process is stationary for $R_X(t+\tau,t)$ depends only on τ.

3) The power spectral density is the Fourier transform of the autocorrelation function, hence

$$\mathcal{S}_X(f) = \frac{\sigma^2}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right].$$

4) If $\sigma_X^2 \neq \sigma_Y^2$, then

$$m_X(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0$$

and

$$\begin{aligned}
R_X(t+\tau,t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) +\\
&\quad E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)\\
&= \frac{\sigma_X^2}{2}[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)] +\\
&\quad \frac{\sigma_Y^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))]\\
&= \frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0(2t+\tau) +\\
&\quad \frac{\sigma_X^2 + \sigma_Y^2}{2}\cos(2\pi f_0\tau)
\end{aligned}$$

The process is not stationary for $R_X(t+\tau,t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$.

Problem 5.39

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = E[Y(t_2)X(t_1)] = R_{YX}(t_2,t_1)$$

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If we let $\tau = t_1 - t_2$, then using the previous result and the fact that $X(t)$, $Y(t)$ are jointly stationary, so that $R_{XY}(t_1, t_2)$ depends only on τ, we obtain

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2) = R_{YX}(t_2 - t_1) = R_{YX}(-\tau)$$

Taking the Fourier transform of both sides of the previous relation we obtain

$$\begin{aligned} \mathcal{S}_{XY}(f) &= \mathcal{F}[R_{XY}(\tau)] = \mathcal{F}[R_{YX}(-\tau)] \\ &= \int_{-\infty}^{\infty} R_{YX}(-\tau) e^{-j2\pi f\tau} d\tau \\ &= \left[\int_{-\infty}^{\infty} R_{YX}(\tau') e^{-j2\pi f\tau'} d\tau'\right]^* = \mathcal{S}_{YX}^*(f) \end{aligned}$$

Problem 5.40

1) $\mathcal{S}_X(f) = \frac{N_0}{2}$, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_X(\tau) \star h(\tau) \star h(-\tau), \qquad \mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$$

With $H(f) = \Pi(\frac{f}{2B})$ we have $|H(f)|^2 = \Pi^2(\frac{f}{2B}) = \Pi(\frac{f}{2B})$ so that

$$\mathcal{S}_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = 2B\frac{N_0}{2}\text{sinc}(2B\tau) = BN_0\text{sinc}(2B\tau)$$

2) The output random process $Y(t)$ is a zero mean Gaussian process with variance

$$\sigma_{Y(t)}^2 = E[Y^2(t)] = E[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian processes $Y(t+\tau)$, $Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{COV(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{E[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$, we have $R_Y(\frac{1}{2B}) = \text{sinc}(1) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau)Y(t)} = \frac{1}{2\pi BN_0} e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated they are also independent.

Problem 5.41

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The impulse response of a delay line that introduces a delay equal to Δ is $h(t) = \delta(t - \Delta)$. The output autocorrelation function is

$$R_Y(\tau) = R_X(\tau) \star h(\tau) \star h(-\tau)$$

But,

$$\begin{aligned} h(\tau) \star h(-\tau) &= \int_{-\infty}^{\infty} \delta(-(t-\Delta))\delta(\tau - (t-\Delta))dt \\ &= \int_{-\infty}^{\infty} \delta(t-\Delta)\delta(\tau - (t-\Delta))dt \\ &= \int_{-\infty}^{\infty} \delta(t')\delta(\tau - t')dt' = \delta(\tau) \end{aligned}$$

Hence,

$$R_Y(\tau) = R_X(\tau) \star \delta(\tau) = R_X(\tau)$$

This is to be expected since a delay line does not alter the spectral characteristics of the input process.

Problem 5.42

The converse of the theorem is not true. Consider for example the random process $X(t) = \cos(2\pi f_0 t) + X$ where X is a random variable. Clearly

$$m_X(t) = \cos(2\pi f_0 t) + m_X$$

is a function of time. However, passing this process through the LTI system with transfer function $\Pi(\frac{f}{2W})$ with $W < f_0$ produces the stationary random process $Y(t) = X$.

Problem 5.43

1) $Y(t) = \frac{d}{dt}X(t)$ can be considered as the output process of a differentiator which is known to be a LTI system with impulse response $h(t) = \delta'(t)$. Since $X(t)$ is stationary, its mean is constant so that

$$m_Y(t) = m_{X'}(t) = [m_X(t)]' = 0$$

To prove that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated we have to prove that $R_{XX'}(0) - m_X(t)m_{X'}(t) = 0$ or since $m_{X'}(t) = 0$ it suffices to prove that $R_{XX'}(0) = 0$. But,

$$R_{XX'}(\tau) = R_X(\tau) \star \delta'(-\tau) = -R_X(\tau) \star \delta'(\tau) = -R'_X(\tau)$$

and since $R_X(\tau) = R_X(-\tau)$ we obtain

$$R_{XX'}(\tau) = -R'_X(\tau) = R'_X(-\tau) = -R_{XX'}(-\tau)$$

Thus $R_{XX'}(\tau)$ is an odd function and its value at the origin should be equal to zero

$$R_{XX'}(0) = 0$$

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The last proves that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated.

2) The autocorrelation function of the sum $Z(t) = X(t) + \frac{d}{dt}X(t)$ is

$$R_Z(\tau) = R_X(\tau) + R_{X'}(\tau) + R_{XX'}(\tau) + R_{X'X}(\tau)$$

If we take the Fourier transform of both sides we obtain

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f) + \mathcal{S}_{X'}(f) + 2\text{Re}[\mathcal{S}_{XX'}(f)]$$

But, $\mathcal{S}_{XX'}(f) = \mathcal{F}[-R_X(\tau) \star \delta'(\tau)] = \mathcal{S}_X(f)(-j2\pi f)$ so that $\text{Re}[\mathcal{S}_{XX'}(f)] = 0$. Thus,

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f) + \mathcal{S}_{X'}(f)$$

3) Since the transfer function of a differentiator is $j2\pi f$, we have $\mathcal{S}_{X'}(f) = 4\pi^2 f^2 \mathcal{S}_X(f)$, hence

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f)(1 + 4\pi^2 f^2)$$

Problem 5.44

1) The impulse response of the system is $h(t) = \mathcal{L}[\delta(t)] = \delta'(t) + \delta'(t - T)$. It is a LTI system so that the output process is a stationary. This is true since $Y(t + c) = \mathcal{L}[X(t + c)]$ for all c, so if $X(t)$ and $X(t + c)$ have the same statistical properties, so do the processes $Y(t)$ and $Y(t + c)$.

2) $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$. But, $H(f) = j2\pi f + j2\pi f e^{-j2\pi f T}$ so that

$$\begin{aligned}
\mathcal{S}_Y(f) &= \mathcal{S}_X(f) 4\pi^2 f^2 \left|1 + e^{-j2\pi f T}\right|^2 \\
&= \mathcal{S}_X(f) 4\pi^2 f^2 [(1 + \cos(2\pi f T))^2 + \sin^2(2\pi f T)] \\
&= \mathcal{S}_X(f) 8\pi^2 f^2 (1 + \cos(2\pi f T))
\end{aligned}$$

3) The frequencies for which $|H(f)|^2 = 0$ will not be present at the output. These frequencies are $f = 0$, for which $f^2 = 0$ and $f = \frac{1}{2T} + \frac{k}{T}$, $k \in \mathcal{Z}$, for which $\cos(2\pi f T) = -1$.

Problem 5.45

1) $Y(t) = X(t) \star (\delta(t) - \delta(t - T))$. Hence,

$$\begin{aligned}
\mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|1 - e^{-j2\pi f T}|^2 \\
&= \mathcal{S}_X(f) 2(1 - \cos(2\pi f T))
\end{aligned}$$

2) $Y(t) = X(t) \star (\delta'(t) - \delta(t))$. Hence,

$$\begin{aligned}
\mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|j2\pi f - 1|^2 \\
&= \mathcal{S}_X(f)(1 + 4\pi^2 f^2)
\end{aligned}$$

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3) $Y(t) = X(t) \star (\delta'(t) - \delta(t-T))$. Hence,

$$\begin{aligned}\mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|j2\pi f - e^{-j2\pi fT}|^2 \\ &= \mathcal{S}_X(f)(1 + 4\pi^2 f^2 + 4\pi f \sin(2\pi fT))\end{aligned}$$

Problem 5.46

Using Schwartz's inequality

$$E^2[X(t+\tau)Y(t)] \leq E[X^2(t+\tau)]E[Y^2(t)] = R_X(0)R_Y(0)$$

where equality holds for independent $X(t)$ and $Y(t)$. Thus

$$|R_{XY}(\tau)| = \left(E^2[X(t+\tau)Y(t)]\right)^{\frac{1}{2}} \leq R_X^{1/2}(0)R_Y^{1/2}(0)$$

The second part of the inequality follows from the fact $2ab \leq a^2 + b^2$. Thus, with $a = R_X^{1/2}(0)$ and $b = R_Y^{1/2}(0)$ we obtain

$$R_X^{1/2}(0)R_Y^{1/2}(0) \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Problem 5.47

1)

$$\begin{aligned}R_{XY}(\tau) &= R_X(\tau) \star \delta(-\tau - \Delta) = R_X(\tau) \star \delta(\tau + \Delta) \\ &= e^{-\alpha|\tau|} \star \delta(\tau + \Delta) = e^{-\alpha|\tau+\Delta|} \\ R_Y(\tau) &= R_{XY}(\tau) \star \delta(\tau - \Delta) = e^{-\alpha|\tau+\Delta|} \star \delta(\tau - \Delta) \\ &= e^{-\alpha|\tau|}\end{aligned}$$

2)

$$\begin{aligned}R_{XY}(\tau) &= e^{-\alpha|\tau|} \star (-\frac{1}{\tau}) = -\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{t-v} dv \\ R_Y(\tau) &= R_{XY}(\tau) \star \frac{1}{\tau} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{s-v}\frac{1}{\tau - s} ds dv\end{aligned} \tag{0.1}$$

The case of $R_Y(\tau)$ can be simplified as follows. Note that $R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_Y(f)]$ where $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$. In our case, $\mathcal{S}_X(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$ and $|H(f)|^2 = \pi^2 \text{sgn}^2(f)$. Since $\mathcal{S}_X(f)$ does not contain any impulses at the origin ($f = 0$) for which $|H(f)|^2 = 0$, we obtain

$$R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_Y(f)] = \pi^2 e^{-\alpha|\tau|}$$

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3) The transfer function is $H(f) = \frac{1}{\alpha + j2\pi f}$. Therefore

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)\frac{1}{\alpha^2 + 4\pi^2 f^2} = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2}$$

Since $\frac{2\alpha}{\alpha^2+4\pi^2 f^2} \Leftrightarrow e^{-\alpha|\tau|}$, applying the differentiation in the frequency domain result we have

$$\frac{d}{df}\frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \Leftrightarrow \frac{2\pi}{j}\tau e^{-\alpha|\tau|}$$

resulting in

$$\frac{(j2\pi f)2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2} \Leftrightarrow -\frac{\tau}{2}e^{-\alpha|\tau|}$$

Now we can apply integration in the time domain result to conclude that

$$\frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2} \Leftrightarrow -\frac{1}{2}\int_{-\infty}^{\tau} ue^{-\alpha|u|}\,du$$

Integration of the right hand side is simple and should be carried out considering $\tau < 0$ and $\tau > 0$ separately. If we do this we will have

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2}\right] = \frac{1}{2\alpha}|\tau|e^{-\alpha|\tau|} + \frac{1}{2\alpha^2}e^{-\alpha|\tau|}$$

For $\mathcal{S}_{XY}(f)$ we have

$$\mathcal{S}_{XY}(f) = \mathcal{S}_X(f)H^*(f) = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)(\alpha - j2\pi f)} = \frac{2\alpha(\alpha + j2\pi f)}{(\alpha^2 + 2\pi^2 f^2)^2}$$

or

$$R_{XY}(\tau) = \mathcal{F}^{-1}\left[\frac{2\alpha^2}{(\alpha^2 + 4\pi^2 f^2)^2}\right] + \mathcal{F}^{-1}\left[\frac{j2\pi f(2\alpha)}{(\alpha^2 + 4\pi^2 f^2)^2}\right]$$

The inverse Fourier transform of the first term we have already found, for the second term we apply the differentiation property of the Fourier transform. We have

$$R_{XY}(\tau) = \frac{1}{2\alpha}e^{-\alpha|\tau|} + \frac{1}{2}|\tau|e^{-\alpha|\tau|} + \frac{d}{d\tau}\left(\frac{1}{2\alpha^2}e^{-\alpha|\tau|} + \frac{1}{2\alpha}|\tau|e^{-\alpha|\tau|}\right)$$

This simplifies to

$$R_{XY}(\tau) = \frac{1}{2}e^{-\alpha|\tau|}\left(\frac{1}{\alpha} + |\tau| - \tau\right)$$

4) The system's transfer function is $H(f) = \frac{-1+j2\pi f}{1+j2\pi f}$. Hence,

$$\begin{aligned}\mathcal{S}_{XY}(f) &= \mathcal{S}_X(f)H^*(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}\frac{-1 - j2\pi f}{1 - j2\pi f} \\ &= \frac{4\alpha}{1-\alpha^2}\frac{1}{1 - j2\pi f} + \frac{\alpha - 1}{1 + \alpha}\frac{1}{\alpha + j2\pi f} + \frac{1 + \alpha}{\alpha - 1}\frac{1}{\alpha - j2\pi f}\end{aligned}$$

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Thus,

$$\begin{aligned} R_{XY}(\tau) &= \mathcal{F}^{-1}[\mathcal{S}_{XY}(f)] \\ &= \frac{4\alpha}{1-\alpha^2}e^{\tau}u_{-1}(-\tau) + \frac{\alpha-1}{1+\alpha}e^{-\alpha\tau}u_{-1}(\tau) + \frac{1+\alpha}{\alpha-1}e^{\alpha\tau}u_{-1}(-\tau) \end{aligned}$$

For the output power spectral density we have $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)\frac{1+4\pi^2 f^2}{1+4\pi^2 f^2} = \mathcal{S}_X(f)$. Hence,

$$R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_X(f)] = e^{-\alpha|\tau|}$$

5) The impulse response of the system is $h(t) = \frac{1}{2T}\Pi(\frac{t}{2T})$. Hence,

$$\begin{aligned} R_{XY}(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{-\tau}{2T}) = e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\ &= \frac{1}{2T}\int_{\tau-T}^{\tau+T} e^{-\alpha|v|}dv \end{aligned}$$

If $\tau \geq T$, then

$$R_{XY}(\tau) = -\frac{1}{2T\alpha}e^{-\alpha v}\Big|_{\tau-T}^{\tau+T} = \frac{1}{2T\alpha}\left(e^{-\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right)$$

If $0 \leq \tau < T$, then

$$\begin{aligned} R_{XY}(\tau) &= \frac{1}{2T}\int_{\tau-T}^{0} e^{\alpha v}dv + \frac{1}{2T}\int_{0}^{\tau+T} e^{-\alpha v}dv \\ &= \frac{1}{2T\alpha}\left(2 - e^{\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right) \end{aligned}$$

The autocorrelation of the output is given by

$$\begin{aligned} R_Y(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\ &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Lambda(\frac{\tau}{2T}) \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|x|}{2T}\right)e^{-\alpha|\tau-x|}dx \end{aligned}$$

If $\tau \geq 2T$, then

$$R_Y(\tau) = \frac{e^{-\alpha\tau}}{2T\alpha^2}\left[e^{2\alpha T} + e^{-2\alpha T} - 2\right]$$

If $0 \leq \tau < 2T$, then

$$R_Y(\tau) = \frac{e^{-2\alpha T}}{4T^2\alpha^2}\left[e^{-\alpha\tau} + e^{\alpha\tau}\right] + \frac{1}{T\alpha} - \frac{\tau}{2T^2\alpha^2} - 2\frac{e^{-\alpha\tau}}{4T^2\alpha^2}$$

Problem 5.48

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Consider the random processes $X(t) = Xe^{j2\pi f_0 t}$ and $Y(t) = Ye^{j2\pi f_0 t}$. Clearly

$$R_{XY}(t+\tau, t) = E[X(t+\tau)Y^*(t)] = E[XY]e^{j2\pi f_0 \tau}$$

However, both $X(t)$ and $Y(t)$ are nonstationary for $E[X(t)] = E[X]e^{j2\pi f_0 t}$ and $E[Y(t)] = E[Y]e^{j2\pi f_0 t}$ are not constant.

Problem 5.49

1)

$$\begin{aligned} E[X(t)] &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} A\cos(2\pi f_0 t + \theta)d\theta \\ &= \frac{4A}{\pi}\sin(2\pi f_0 t + \theta)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{4A}{\pi}[\sin(2\pi f_0 t + \frac{\pi}{4}) - \sin(2\pi f_0 t)] \end{aligned}$$

Thus, $E[X(t)]$ is periodic with period $T = \frac{1}{f_0}$.

$$\begin{aligned} R_X(t+\tau, t) &= E[A^2\cos(2\pi f_0(t+\tau)+\Theta)\cos(2\pi f_0 t + \Theta)] \\ &= \frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau)+\Theta) + \cos(2\pi f_0\tau)] \\ &= \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau)+\Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{2}\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\cos(2\pi f_0(2t+\tau)+\theta)d\theta \\ &= \frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{\pi}(\cos(2\pi f_0(2t+\tau)) - \sin(2\pi f_0(2t+\tau))) \end{aligned}$$

which is periodic with period $T' = \frac{1}{2f_0}$. Thus the process is cyclostationary with period $T = \frac{1}{f_0}$. Using the results of Problem 4.48 we obtain

$$\begin{aligned} \mathcal{S}_X(f) &= \mathcal{F}[\frac{1}{T}\int_0^T R_X(t+\tau, t)dt] \\ &= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0\tau) + \frac{A^2}{T\pi}\int_0^T(\cos(2\pi f_0(2t+\tau)) - \sin(2\pi f_0(2t+\tau))dt\right] \\ &= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0\tau)\right] \\ &= \frac{A^2}{4}(\delta(f-f_0) + \delta(f+f_0)) \end{aligned}$$

2)

$$\begin{aligned} R_X(t+\tau, t) &= E[X(t+\tau)X(t)] = E[(X+Y)(X+Y)] \\ &= E[X^2] + E[Y^2] + E[YX] + E[XY] \\ &= E[X^2] + E[Y^2] + 2E[X][Y] \end{aligned}$$

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where the last equality follows from the independence of X and Y. But, $E[X] = 0$ since X is uniform on $[-1,\ 1]$ so that

$$R_X(t+\tau, t) = E[X^2] + E[Y^2] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

The Fourier transform of $R_X(t+\tau, t)$ is the power spectral density of $X(t)$. Thus

$$\mathcal{S}_X(f) = \mathcal{F}[R_X(t+\tau, t)] = \frac{2}{3}\delta(f)$$

Problem 5.50

$h(t) = e^{-\beta t}u_{-1}(t) \Rightarrow H(f) = \frac{1}{\beta + j2\pi f}$. The power spectral density of the input process is $\mathcal{S}_X(f) = \mathcal{F}[e^{-\alpha|\tau|}] = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$. If $\alpha = \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2}$$

If $\alpha \neq \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)(\beta^2 + 4\pi^2 f^2)}$$

Problem 5.51

1) Let $Y(t) = X(t) + N(t)$. The process $\hat{X}(t)$ is the response of the system $h(t)$ to the input process $Y(t)$ so that

$$\begin{aligned} R_{Y\hat{X}}(\tau) &= R_Y(\tau) \star h(-\tau) \\ &= [R_X(\tau) + R_N(\tau) + R_{XN}(\tau) + R_{NX}(\tau)] \star h(-\tau) \end{aligned}$$

Also by definition

$$\begin{aligned} R_{Y\hat{X}}(\tau) &= E[(X(t+\tau) + N(t+\tau))\hat{X}(t)] = R_{X\hat{X}}(\tau) + R_{N\hat{X}}(\tau) \\ &= R_{X\hat{X}}(\tau) + R_N(\tau) \star h(-\tau) + R_{NX}(\tau) \star h(-\tau) \end{aligned}$$

Substituting this expression for $R_{Y\hat{X}}(\tau)$ in the previous one, and cancelling common terms we obtain

$$R_{X\hat{X}}(\tau) = R_X(\tau) \star h(-\tau) + R_{XN}(\tau) \star h(-\tau)$$

2)

$$E\left[(X(t) - \hat{X}(t))^2\right] = R_X(0) + R_{\hat{X}}(0) - R_{X\hat{X}}(0) - R_{\hat{X}X}(0)$$

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We can write $E\left[(X(t)-\hat{X}(t))^2\right]$ in terms of the spectral densities as

$$\begin{aligned}
E\left[(X(t)-\hat{X}(t))^2\right] &= \int_{-\infty}^{\infty}(\mathcal{S}_X(f)+\mathcal{S}_{\hat{X}}(f)-2\mathcal{S}_{X\hat{X}}(f))df \\
&= \int_{-\infty}^{\infty}\Big[\mathcal{S}_X(f)+(\mathcal{S}_X(f)+\mathcal{S}_N(f)+2\mathrm{Re}[\mathcal{S}_{XN}(f)])|H(f)|^2 \\
&\quad -2(\mathcal{S}_X(f)+\mathcal{S}_{XN}(f))H^*(f)\Big]df
\end{aligned}$$

To find the $H(f)$ that minimizes $E\left[(X(t)-\hat{X}(t))^2\right]$ we set the derivative of the previous expression, with respect to $H(f)$, to zero. By doing so we obtain

$$H(f)=\frac{\mathcal{S}_X(f)+\mathcal{S}_{XN}(f)}{\mathcal{S}_X(f)+\mathcal{S}_N(f)+2\mathrm{Re}[\mathcal{S}_{XN}(f)]}$$

3) If $X(t)$ and $N(t)$ are independent, then

$$R_{XN}(\tau)=E[X(t+\tau)N(t)]=E[X(t+\tau)]E[N(t)]$$

Since $E[N(t)]=0$ we obtain $R_{XN}(\tau)=0$ and the optimum filter is

$$H(f)=\frac{\mathcal{S}_X(f)}{\mathcal{S}_X(f)+\frac{N_0}{2}}$$

The corresponding value of $E\left[(X(t)-\hat{X}(t))^2\right]$ is

$$E_{\min}\left[(X(t)-\hat{X}(t))^2\right]=\int_{-\infty}^{\infty}\frac{\mathcal{S}_X(f)N_0}{2\mathcal{S}_X(f)+N_0}df$$

4) With $\mathcal{S}_N(f)=1$, $\mathcal{S}_X(f)=\frac{1}{1+f^2}$ and $\mathcal{S}_{XN}(f)=0$, then

$$H(f)=\frac{\frac{1}{1+f^2}}{1+\frac{1}{1+f^2}}=\frac{1}{2+f^2}$$

Problem 5.52

1) Let $\hat{X}(t)$ and $\tilde{X}(t)$ be the outputs of the systems $h(t)$ and $g(t)$ when the input $Z(t)$ is applied. Then,

$$\begin{aligned}
E[(X(t)-\tilde{X}(t))^2] &= E[(X(t)-\hat{X}(t)+\hat{X}(t)-\tilde{X}(t))^2] \\
&= E[(X(t)-\hat{X}(t))^2]+E[(\hat{X}(t)-\tilde{X}(t))^2] \\
&\quad +E[(X(t)-\hat{X}(t))\cdot(\hat{X}(t)-\tilde{X}(t))]
\end{aligned}$$

But,

$$\begin{aligned}
&E[(X(t)-\hat{X}(t))\cdot(\hat{X}(t)-\tilde{X}(t))] \\
&= E[(X(t)-\hat{X}(t))\cdot Z(t)\star(h(t)-g(t))] \\
&= E\left[(X(t)-\hat{X}(t))\int_{-\infty}^{\infty}(h(\tau)-g(\tau))Z(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty}E\left[(X(t)-\hat{X}(t))Z(t-\tau)\right](h(\tau)-g(\tau))d\tau=0
\end{aligned}$$

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where the last equality follows from the assumption $E\left[(X(t)-\hat{X}(t))Z(t-\tau)\right]=0$ for all t,τ. Thus,

$$E[(X(t)-\tilde{X}(t))^2]=E[(X(t)-\hat{X}(t))^2]+E[(\hat{X}(t)-\tilde{X}(t))^2]$$

and this proves that

$$E[(X(t)-\hat{X}(t))^2]\leq E[(X(t)-\tilde{X}(t))^2]$$

2)

$$E[(X(t)-\hat{X}(t))Z(t-\tau)]=0\Rightarrow E[X(t)Z(t-\tau)]=E[\hat{X}(t)Z(t-\tau)]$$

or in terms of crosscorrelation functions $R_{XZ}(\tau)=R_{\hat{X}Z}(\tau)=R_{Z\hat{X}}(-\tau)$. However, $R_{Z\hat{X}}(-\tau)=R_Z(-\tau)\star h(\tau)$ so that

$$R_{XZ}(\tau)=R_Z(-\tau)\star h(\tau)=R_Z(\tau)\star h(\tau)$$

3) Taking the Fourier of both sides of the previous equation we obtain

$$\mathcal{S}_{XZ}(f)=\mathcal{S}_Z(f)H(f)\quad\text{or}\quad H(f)=\frac{\mathcal{S}_{XZ}(f)}{\mathcal{S}_Z(f)}$$

4)

$$\begin{aligned}
E[\epsilon^2(t)] &= E\left[(X(t)-\hat{X}(t))((X(t)-\hat{X}(t))\right]\\
&= E[X(t)X(t)]-E[\hat{X}(t)X(t)]\\
&= R_X(0)-E\left[\int_{-\infty}^{\infty}Z(t-v)h(v)X(t)dv\right]\\
&= R_X(0)-\int_{-\infty}^{\infty}R_{ZX}(-v)h(v)dv\\
&= R_X(0)-\int_{-\infty}^{\infty}R_{XZ}(v)h(v)dv
\end{aligned}$$

where we have used the fact that $E[(X(t)-\hat{X}(t))\hat{X}(t)]=E[(X(t)-\hat{X}(t))Z(t)\star h(t)]=0$

Problem 5.53

the noise equivalent bandwidth of a filter is

$$B_{neq}=\frac{\int_{-\infty}^{\infty}|H(f)|^2df}{2H_{\max}^2}$$

If we have an ideal bandpass filter of bandwidth W, then $H(f)=1$ for $|f-f_0|<W$ where f_0 is the central frequency of the filter. Hence,

$$B_{neq}=\frac{1}{2}\left[\int_{-f_0-\frac{W}{2}}^{-f_0+\frac{W}{2}}df+\int_{f_0-\frac{W}{2}}^{f_0+\frac{W}{2}}df\right]=W$$

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Problem 5.54

1) The power spectral density of the in-phase and quadrature components is given by

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} \mathcal{S}_n(f - f_0) + \mathcal{S}_n(f + f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

If the passband of the ideal filter extends from 3 to 11 KHz, then f_0 =7 KHz is the mid-band frequency so that

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} N_0 & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$\mathcal{S}_{n_c n_s}(f) = \begin{cases} j[\mathcal{S}_n(f + f_0) - \mathcal{S}_n(f - f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

However $\mathcal{S}_n(f + f_0) = \mathcal{S}_n(f - f_0)$ for $|f| < 7$ and therefore $\mathcal{S}_{n_c n_s}(f) = 0$. It turns then that the crosscorrelation $R_{n_c n_s}(\tau)$ is zero.

2) With f_0=6 KHz

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} \frac{N_0}{2} & 3 < |f| < 5 \\ N_0 & |f| < 3 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$\mathcal{S}_{n_c n_s}(f) = \begin{cases} -j\frac{N_0}{2} & -5 < f < 3 \\ j\frac{N_0}{2} & 3 < f < 5 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} R_{n_c n_s}(\tau) &= \mathcal{F}^{-1}\left[-j\frac{N_0}{2}\Pi(\frac{t+4}{2}) + j\frac{N_0}{2}\Pi(\frac{t-4}{2})\right] \\ &= -j\frac{N_0}{2}2\text{sinc}(2\tau)e^{-j2\pi 4\tau} + j\frac{N_0}{2}2\text{sinc}(2\tau)e^{j2\pi 4\tau} \\ &= -2N_0\text{sinc}(2\tau)\sin(2\pi 4\tau) \end{aligned}$$

Problem 5.55

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The in-phase component of $X(t)$ is

$$\begin{aligned}
X_c(t) &= X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t) \\
&= \sum_{n=-\infty}^{\infty} A_n p(t-nT)\cos(2\pi f_0(t-nT)) \\
&\quad + \sum_{n=-\infty}^{\infty} A_n \hat{p}(t-nT)\sin(2\pi f_0(t-nT)) \\
&= \sum_{n=-\infty}^{\infty} A_n \left(p(t-nT)\cos(2\pi f_0(t-nT)) + \hat{p}(t-nT)\sin(2\pi f_0(t-nT))\right) \\
&= \sum_{n=-\infty}^{\infty} A_n p_c(t-nT)
\end{aligned}$$

where we have used the fact $p_c(t) = p(t)\cos(2\pi f_0 t) + \hat{p}(t)\sin(2\pi f_0 t)$. Similarly for the quadrature component

$$\begin{aligned}
X_s(t) &= \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t) \\
&= \sum_{n=-\infty}^{\infty} A_n \hat{p}(t-nT)\cos(2\pi f_0(t-nT)) \\
&\quad - \sum_{n=-\infty}^{\infty} A_n p(t-nT)\sin(2\pi f_0(t-nT)) \\
&= \sum_{n=-\infty}^{\infty} A_n \left(\hat{p}(t-nT)\cos(2\pi f_0(t-nT)) - p(t-nT)\sin(2\pi f_0(t-nT))\right) \\
&= \sum_{n=-\infty}^{\infty} A_n p_s(t-nT)
\end{aligned}$$

Problem 5.56

The envelope $V(t)$ of a bandpass process is defined to be

$$V(t) = \sqrt{X_c^2(t) + X_s^2(t)}$$

where $X_c(t)$ and $X_s(t)$ are the in-phase and quadrature components of $X(t)$ respectively. However, both the in-phase and quadrature components are lowpass processes and this makes $V(t)$ a lowpass process independent of the choice of the center frequency f_0.

Problem 5.57

1) The power spectrum of the bandpass signal is

$$\mathcal{S}_n(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$

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Hence,

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

The power content of the in-phase and quadrature components of $n(t)$ is $P_n = \int_{-W}^{W} N_0 df = 2N_0 W$

2) Since $\mathcal{S}_{n_c n_s}(f) = 0$, the processes $N_c(t)$, $N_s(t)$ are independent zero-mean Gaussian with variance $\sigma^2 = P_n = 2N_0 W$. Hence, $V(t) = \sqrt{N_c^2(t) + N_s^2(t)}$ is Rayleigh distributed and the PDF is given by

$$f_V(v) = \begin{cases} \frac{v^2}{2N_0 W} e^{-\frac{v^2}{4N_0 W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

3) $X(t)$ is given by

$$X(t) = (A + N_c(t))\cos(2\pi f_0 t) - N_S(t)\sin(2\pi f_0 t)$$

The process $A + N_c(t)$ is Gaussian with mean A and variance $2N_0 W$. Hence, $V(t) = \sqrt{(A + N_c(t))^2 + N_s^2(t)}$ follows the Rician distribution (see Problem 4.31). The density function of the envelope is given by

$$f_V(v) = \begin{cases} \frac{v}{2N_0 W} I_0(\frac{Av}{2N_0 W}) e^{-\frac{v^2 + A^2}{4N_0 W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

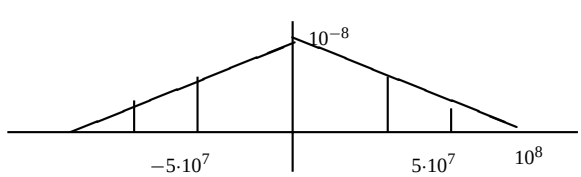
where

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos u} du$$

Problem 5.58

1) The power spectral density $\mathcal{S}_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$\begin{aligned} P &= \int_{-51\times10^6}^{-49\times10^6} 10^{-8}\left(1 + \frac{f}{10^8}\right) df + \int_{49\times10^6}^{51\times10^6} 10^{-8}\left(1 - \frac{f}{10^8}\right) df \\ &= 10^{-8} x \Big|_{-51\times10^6}^{-49\times10^6} + 10^{-16}\frac{1}{2}x^2\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-8} x\Big|_{49\times10^6}^{51\times10^6} - 10^{-16}\frac{1}{2}x^2\Big|_{49\times10^6}^{51\times10^6} \\ &= 2 \times 10^{-2} \end{aligned}$$



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2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50 \times 10^6 t) - N_s(t)\sin(2\pi 50 \times 10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t)\cos(2\pi 50 \times 10^6 t) + \hat{N}(t)\sin(2\pi 50 \times 10^6 t) \\ N_s(t) &= \hat{N}(t)\cos(2\pi 50 \times 10^6 t) - N(t)\sin(2\pi 50 \times 10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi 50 \times 10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi 50 \times 10^6 t) \\ &= E[|N(t)|^2] = 2 \times 10^{-2} \end{aligned}$$

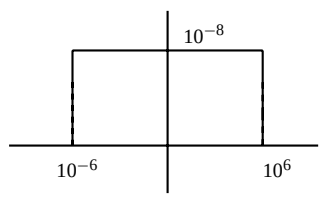
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2 \times 10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$\mathcal{S}_{N_c}(f) = \mathcal{S}_{N_s}(f) = \begin{cases} \mathcal{S}_N(f - 50 \times 10^6) + \mathcal{S}_N(f + 50 \times 10^6) & |f| \leq 50 \times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}_{N_c}(f)$ is depicted in the next figure. The power content of $\mathcal{S}_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2 \times 10^{-2}$$



4) The power spectral density of the output is given by

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = (|f| - 49 \times 10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49 \times 10^6 \leq |f| \leq 51 \times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= \int_{-51\times 10^6}^{-49\times 10^6} (-f - 49 \times 10^6)(10^{-8} + 10^{-16} f) df \\ &\quad + \int_{49\times 10^6}^{51\times 10^6} (f - 49 \times 10^6)(10^{-8} - 10^{-16} f) df \\ &= 2 \times 10^4 - \frac{4}{3} 10^2 \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} \mathcal{S}_{Y_c}(f) = \mathcal{S}_{Y_s}(f) &= \left((f + 50 \times 10^6) - 49 \times 10^6\right)\left(10^{-8} - 10^{-16}(f + 50 \times 10^6)\right) \\ &\quad + \left(-(f - 50 \times 10^6) - 49 \times 10^6\right)\left(10^{-8} + 10^{-16}(f - 50 \times 10^6)\right) \\ &= -2 \times 10^{-16} f^2 + 10^{-2} \end{aligned}$$

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for $|f| \leq 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned} P_{Y_c} = P_{Y_s} &= \int_{-10^6}^{10^6} (-2 \times 10^{-16} f^2 + 10^{-2}) df \\ &= -2 \times 10^{-16} \frac{1}{3} f^3 \Big|_{-10^6}^{10^6} + 10^{-2} f \Big|_{-10^6}^{10^6} \\ &= 2 \times 10^4 - \frac{4}{3} 10^2 = P_Y \end{aligned}$$

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