Chapter 2

Signal and Linear System Theory

2.1 Problem Solutions

Problem 2.1
For the single-sided spectra, write the signal in terms of cosines:

$$\begin{aligned} x(t) &= 10\cos(4\pi t + \pi/8) + 6\sin(8\pi t + 3\pi/4) \\ &= 10\cos(4\pi t + \pi/8) + 6\cos(8\pi t + 3\pi/4 - \pi/2) \\ &= 10\cos(4\pi t + \pi/8) + 6\cos(8\pi t + \pi/4) \end{aligned}$$

For the double-sided spectra, write the signal in terms of complex exponentials using Euler's theorem:

$$\begin{aligned} x(t) &= 5\exp[(4\pi t + \pi/8)] + 5\exp[-j(4\pi t + \pi/8)] \\ &\quad +3\exp[j(8\pi t + 3\pi/4)] + 3\exp[-j(8\pi t + 3\pi/4)] \end{aligned}$$

The two sets of spectra are plotted in Figures 2.1 and 2.2.

Problem 2.2
The result is

$$\begin{aligned} x(t) &= 4e^{j(8\pi t+\pi/2)} + 4e^{-j(8\pi t+\pi/2)} + 2e^{j(4\pi t-\pi/4)} + 2e^{-j(4\pi t-\pi/4)} \\ &= 8\cos(8\pi t + \pi/2) + 4\cos(4\pi t - \pi/4) \\ &= -8\sin(8\pi t) + 4\cos(4\pi t - \pi/4) \end{aligned}$$

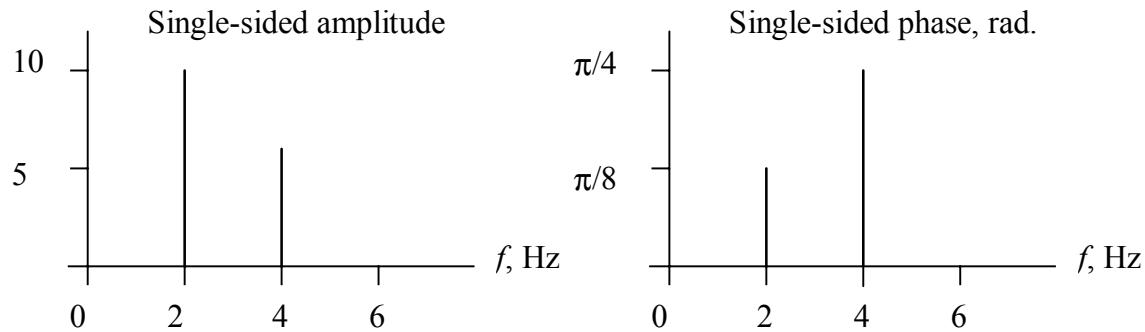

Figure 2.1:

Problem 2.3
(a) Not periodic.
(b) Periodic. To find the period, note that

$$\frac{6\pi}{2\pi} = n_1 f_0 \text{ and } \frac{20\pi}{2\pi} = n_2 f_0$$

Therefore

$$\frac{10}{3} = \frac{n_2}{n_1}$$

Hence, take $n_1 = 3$, $n_2 = 10$, and $f_0 = 1$ Hz.
(c) Periodic. Using a similar procedure as used in (b), we find that $n_1 = 2$, $n_2 = 7$, and $f_0 = 1$ Hz.
(d) Periodic. Using a similar procedure as used in (b), we find that $n_1 = 2$, $n_2 = 3$, $n_3 = 11$, and $f_0 = 1$ Hz.

Problem 2.4
(a) The single-sided amplitude spectrum consists of a single line of height 5 at frequency 6 Hz, and the phase spectrum consists of a single line of height -$\pi/6$ radians at frequency 6 Hz. The double-sided amplitude spectrum consists of lines of height 2.5 at frequencies of 6 and -6 Hz, and the double-sided phase spectrum consists of a line of height -$\pi/6$ radians at frequency 6 Hz and a line of height $\pi/6$ at frequency -6 radians Hz.
(b) Write the signal as

$$x_b(t) = 3\cos(12\pi t - \pi/2) + 4\cos(16\pi t)$$

From this it is seen that the single-sided amplitude spectrum consists of lines of height 3 and 4 at frequencies 6 and 8 Hz, respectively, and the single-sided phase spectrum consists

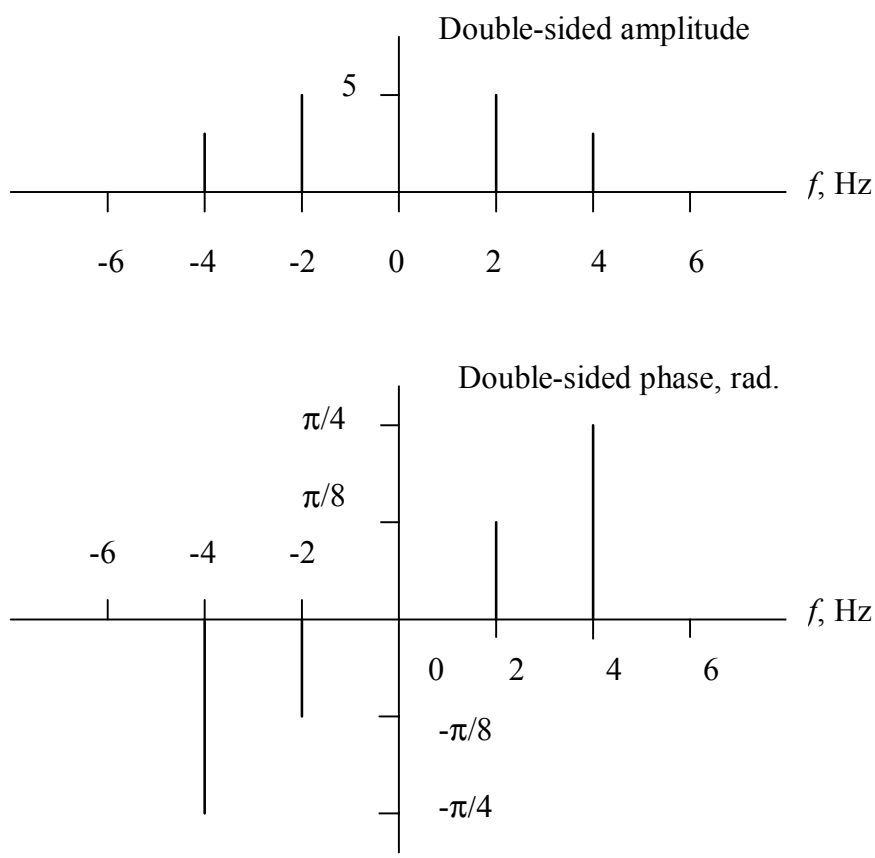
Double-sided amplitude
5
f, Hz
-6
-4
-2
0
2
4
6
Double-sided phase, rad.
π/4
π/8
-6
-4
-2
f, Hz
0
2
4
6
-π/8
-π/4

Figure 2.2:

of a line of height -$\pi/2$ radians at frequency 6 Hz. The double-sided amplitude spectrum consists of lines of height 1.5 and 2 at frequencies of 6 and 8 Hz, respectively, and lines of height 1.5 and 2 at frequencies -6 and -8 Hz, respectively. The double-sided phase spectrum consists of a line of height -$\pi/2$ radians at frequency 6 Hz and a line of height $\pi/2$ radians at frequency -6 Hz.

Problem 2.5
(a) This function has area

$$\begin{aligned}\text{Area} &= \int_{-\infty}^{\infty} \epsilon^{-1}\left[\frac{\sin(\pi t/\epsilon)}{(\pi t/\epsilon)}\right]^2 dt \\ &= \int_{-\infty}^{\infty}\left[\frac{\sin(\pi u)}{(\pi u)}\right]^2 du = 1\end{aligned}$$

A sketch shows that no matter how small ϵ is, the area is still 1. With $\epsilon \to 0$, the central lobe of the function becomes narrower and higher. Thus, in the limit, it approximates a delta function.
(b) The area for the function is

$$\text{Area} = \int_{-\infty}^{\infty} \frac{1}{\epsilon}\exp(-t/\epsilon)u(t)\,dt = \int_{0}^{\infty}\exp(-u)du = 1$$

A sketch shows that no matter how small ϵ is, the area is still 1. With $\epsilon \to 0$, the function becomes narrower and higher. Thus, in the limit, it approximates a delta function.
(c) Area $= \int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon}\left(1 - |t|/\epsilon\right) dt = \int_{-1}^{1} \Lambda(t)\,dt = 1$. As $\epsilon \to 0$, the function becomes narrower and higher, so it approximates a delta function in the limit.

Problem 2.6
(a) 513; (b) 183; (c) 0; (d) 95,583.8; (e) -157.9.

Problem 2.7
(a), (c), (e), and (f) are periodic. Their periods are 1 s, 4 s, 3 s, and 2/7 s, respectively. The waveform of part (c) is a periodic train of impulses extending from -∞ to ∞ spaced by 4 s. The waveform of part (a) is a complex sum of sinusoids that repeats (plot). The waveform of part (e) is a doubly-infinite train of square pulses, each of which is one unit high and one unit wide, centered at $\cdots,\ -6,\ -3,\ 0,\ 3,\ 6,\ \cdots$. Waveform (f) is a raised cosine of minimum and maximum amplitudes 0 and 2, respectively.

Problem 2.8
(a) The result is

$$x(t) = \mathrm{Re}\left(e^{j6\pi t}\right) + 6\,\mathrm{Re}\left(e^{j(12\pi t - \pi/2)}\right) = \mathrm{Re}\left[e^{j6\pi t} + 6e^{j(12\pi t - \pi/2)}\right]$$

(b) The result is

$$x(t) = \frac{1}{2}e^{j6\pi t} + \frac{1}{2}e^{-j6\pi t} + 3e^{j(12\pi t - \pi/2)} + 3e^{-j(12\pi t - \pi/2)}$$

(c) The single-sided amplitude spectrum consists of lines of height 1 and 6 at frequencies of 3 and 6 Hz, respectively. The single-sided phase spectrum consists of a line of height $-\pi/2$ at frequency 6 Hz. The double-sided amplitude spectrum consists of lines of height 3, 1/2, 1/2, and 3 at frequencies of -6, -3, 3, and 6 Hz, respectively. The double-sided phase spectrum consists of lines of height $\pi/2$ and $-\pi/2$ at frequencies of -6 and 6 Hz, respectively.

Problem 2.9
(a) Power. Since it is a periodic signal, we obtain

$$P_1 = \frac{1}{T_0}\int_0^{T_0} 4\sin^2\left(8\pi t + \pi/4\right) dt = \frac{1}{T_0}\int_0^{T_0} 2\left[1 - \cos\left(16\pi t + \pi/2\right)\right] dt = 2 \text{ W}$$

where $T_0 = 1/8$ s is the period.
(b) Energy. The energy is

$$E_2 = \int_{-\infty}^{\infty} e^{-2\alpha t} u^2(t) dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}$$

(c) Energy. The energy is

$$E_3 = \int_{-\infty}^{\infty} e^{2\alpha t} u^2(-t) dt = \int_{-\infty}^{0} e^{2\alpha t} dt = \frac{1}{2\alpha}$$

(d) Neither energy or power.

$$E_4 = \lim_{T\to\infty}\int_{-T}^{T} \frac{dt}{\left(\alpha^2 + t^2\right)^{1/4}} = \infty$$

$P_4 = 0$ since $\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \frac{dt}{(\alpha^2+t^2)^{1/4}} = 0$.(e) Energy. Since it is the sum of $x_1(t)$ and $x_2(t)$, its energy is the sum of the energies of these two signals, or $E_5 = 1/\alpha$.

(f) Power. Since it is an aperiodic signal (the sine starts at $t = 0$), we use

$$\begin{aligned} P_6 &= \lim_{T\to\infty} \frac{1}{2T} \int_0^T \sin^2(5\pi t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_0^T \frac{1}{2}\left[1 - \cos(10\pi t)\right] dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \left[\frac{T}{2} - \frac{1}{2}\frac{\sin(20\pi t)}{20\pi}\right]_0^T = \frac{1}{4}\ \mathrm{W} \end{aligned}$$

Problem 2.10

(a) Power. Since the signal is periodic with period π/ω, we have

$$P = \frac{\omega}{\pi} \int_0^{\pi/\omega} A^2 \left|\sin(\omega t + \theta)\right|^2 dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} \frac{A^2}{2} \left\{1 - \cos\left[2(\omega t + \theta)\right]\right\} dt = \frac{A^2}{2}$$

(b) Neither. The energy calculation gives

$$E = \lim_{T\to\infty} \int_{-T}^{T} \frac{(A\tau)^2\, dt}{\sqrt{\tau + jt}\sqrt{\tau - jt}} = \lim_{T\to\infty} \int_{-T}^{T} \frac{(A\tau)^2\, dt}{\sqrt{\tau^2 + t^2}} \to \infty$$

The power calculation gives

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{(A\tau)^2\, dt}{\sqrt{\tau^2 + t^2}} = \lim_{T\to\infty} \frac{(A\tau)^2}{2T} \ln\left(\frac{1 + \sqrt{1 + T^2/\tau^2}}{-1 + \sqrt{1 + T^2/\tau^2}}\right) = 0$$

(c) Energy:

$$E = \int_0^\infty A^2 t^4 \exp(-2t/\tau)\, dt = \frac{3}{4} A^2 \tau^5 \quad \text{(use table of integrals)}$$

(d) Energy:

$$E = 2\left(\int_0^{\tau/2} 2^2 dt + \int_{\tau/2}^{\tau} 1^2 dt\right) = 5\tau$$

Problem 2.11

(a) This is a periodic train of "boxcars", each 3 units in width and centered at multiples of 6:

$$P_a = \frac{1}{6} \int_{-3}^{3} \Pi^2\left(\frac{t}{3}\right) dt = \frac{1}{6} \int_{-1.5}^{1.5} dt = \frac{1}{2}\ \mathrm{W}$$

(b) This is a periodic train of unit-high isoceles triangles, each 4 units wide and centered at multiples of 5:

$$P_b = \frac{1}{5}\int_{-2.5}^{2.5} \Lambda^2\left(\frac{t}{2}\right) dt = \frac{2}{5}\int_0^2 \left(1-\frac{t}{2}\right)^2 dt = -\frac{2}{5}\frac{2}{3}\left(1-\frac{t}{2}\right)^3\Bigg|_0^2 = \frac{4}{15}\ \text{W}$$

(c) This is a backward train of sawtooths (right triangles with the right angle on the left), each 2 units wide and spaced by 3 units:

$$P_c = \frac{1}{3}\int_0^2 \left(1-\frac{t}{2}\right)^2 dt = -\frac{1}{3}\frac{2}{3}\left(1-\frac{t}{2}\right)^3\Bigg|_0^2 = \frac{2}{9}\ \text{W}$$

(d) This is a full-wave rectified cosine wave of period 1/5 (the width of each cosine pulse):

$$P_d = 5\int_{-1/10}^{1/10} \left|\cos(5\pi t)\right|^2 dt = 2\times 5\int_0^{1/10}\left[\frac{1}{2}+\frac{1}{2}\cos(10\pi t)\right] dt = \frac{1}{2}\ \text{W}$$

Problem 2.12
(a) $E=\infty,\ P=\infty$; (b) $E=5$ J, $P=0$ W; (c) $E=\infty,\ P=49$ W; (d) $E=\infty$, $P=2$ W.

Problem 2.13
(a) The energy is

$$E = \int_{-6}^{6}\cos^2(6\pi t)\, dt = 2\int_0^6\left[\frac{1}{2}+\frac{1}{2}\cos(12\pi t)\right] dt = 6\ \text{J}$$

(b) The energy is

$$E = \int_{-\infty}^{\infty}\left[e^{-|t|/3}\cos(12\pi t)\right]^2 dt = 2\int_0^{\infty} e^{-2t/3}\left[\frac{1}{2}+\frac{1}{2}\cos(24\pi t)\right] dt$$

where the last integral follows by the eveness of the integrand of the first one. Use a table of definte integrals to obtain

$$E = \int_0^{\infty} e^{-2t/3}dt + \int_0^{\infty} e^{-2t/3}\cos(24\pi t)\, dt = \frac{3}{2}+\frac{2/3}{(2/3)^2+(24\pi)^2}$$

Since the result is finite, this is an energy signal.
(c) The energy is

$$E = \int_{-\infty}^{\infty}\left\{2\left[u(t)-u(t-7)\right]\right\}^2 dt = \int_0^7 4dt = 28\ \text{J}$$

Since the result is finite, this is an energy signal.
(d) Note that

$$\int_{-\infty}^{t} u(\lambda)\, d\lambda = r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

which is called the unit ramp. The energy is

$$E = \int_{-\infty}^{\infty} [r(t) - 2r(t-10) + r(t-20)]^2\, dt = 2\int_0^{10} \left(\frac{t}{10}\right)^2 dt = \frac{20}{3} \text{ J}$$

where the last integral follows because the integrand is a symmetrical triangle about $t = 10$. Since the result is finite, this is an energy signal.

Problem 2.14
(a) Expand the integrand, integrate term by term, and simplify making use of the orthogonality property of the orthonormal functions.
(b) Add and subtract the quantity suggested right above (2.34) and simplify.
(c) These are unit-high rectangular pulses of width $T/4$. They are centered at $t = T/8,\ 3T/8,\ 5T/8$, and $7T/8$. Since they are spaced by $T/4$, they are adjacent to each other and fill the interval $[0,\ T]$.
(d) Using the expression for the generalized Fourier series coefficients, we find that $X_1 = 1/8,\ X_2 = 3/8,\ X_3 = 5/8$, and $X_4 = 7/8$. Also, $c_n = T/4$. Thus, the ramp signal is approximated by

$$\frac{t}{T} = \frac{1}{8}\phi_1(t) + \frac{3}{8}\phi_2(t) + \frac{5}{8}\phi_3(t) + \frac{7}{8}\phi_4(t),\ 0 \leq t \leq T$$

where the $\phi_n(t)$s are given in part (c).
(e) These are unit-high rectangular pulses of width $T/2$ and centered at $t = T/4$ and $3T/4$. We find that $X_1 = 1/4$ and $X_2 = 3/4$.
(f) To compute the ISE, we use

$$\epsilon_N = \int_T |x(t)|^2\, dt - \sum_{n=1}^{N} c_n \left|X_n^2\right|$$

Note that $\int_T |x(t)|^2\, dt = \int_0^T (t/T)^2\, dt = T/3$. Hence, for (d),
$\text{ISE}_\text{d} = \frac{T}{3} - \frac{T}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right) = 5.208 \times 10^{-3} T$.
For (e), $\text{ISE}_\text{e} = \frac{T}{3} - \frac{T}{2}\left(\frac{1}{16} + \frac{9}{16}\right) = 2.083 \times 10^{-2} T$.

Problem 2.15
(a) The Fourier coefficients are (note that the period $= \frac{1}{2}\frac{2\pi}{\omega_0}$)

$$X_{-1} = X_1 = \frac{1}{4};\ X_0 = \frac{1}{2}$$

All other coefficients are zero.
(b) The Fourier coefficients for this case are

$$X_{-1} = X_1^* = \frac{1}{2}(1+j)$$

All other coefficients are zero.
(c) The Fourier coefficients for this case are (note that the period is $\frac{2\pi}{2\omega_0}$)

$$X_{-2} = X_2 = \frac{1}{8};\ X_{-1} = X_1 = \frac{1}{4};\ X_0 = -\frac{1}{4}$$

All other coefficients are zero.
(d) The Fourier coefficients for this case are

$$X_{-3} = X_3 = \frac{1}{8};\ X_{-1} = X_1 = \frac{3}{8}$$

All other coefficients are zero.

Problem 2.16
The expansion interval is $T_0 = 4$ and the Fourier coefficients are

$$X_n = \frac{1}{4}\int_{-2}^{2} 2t^2 e^{-jn(\pi/2)t} dt = \frac{2}{4}\int_0^2 2t^2 \cos\left(\frac{n\pi t}{2}\right) dt$$

which follows by the eveness of the integrand. Let $u = n\pi t/2$ to obtain the form

$$X_n = 2\left(\frac{2}{n\pi}\right)^3 \int_0^{n\pi} u^2 \cos u\ du = \frac{16}{(n\pi)^2}(-1)^n$$

If n is odd, the Fourier coefficients are zero as is evident from the eveness of the function being represented. If $n = 0$, the integral for the coefficients is

$$X_0 = \frac{1}{4}\int_{-2}^{2} 2t^2 dt = \frac{8}{3}$$

The Fourier series is therefore

$$x(t) = \frac{8}{3} + \sum_{n=-\infty,\, n\neq 0}^{\infty} (-1)^n \frac{16}{n\pi} e^{jn(\pi/2)t}$$

Problem 2.17

Parts (a) through (c) were discussed in the text. For (d), break the integral for $x(t)$ up into a part for $t < 0$ and a part for $t > 0$. Then use the odd half-wave symmetry contition.

Problem 2.18

This is a matter of integration. Only the solution for part (b) will be given here. The integral for the Fourier coefficients is (note that the period really is $T_0/2$)

$$\begin{aligned} X_n &= \frac{A}{T_0}\int_0^{T_0/2} \sin(\omega_0 t)\, e^{-jn\omega_0 t} dt \\ &= -\frac{Ae^{-jn\omega_0 t}}{\omega_0 T_0 (1-n^2)} \left[jn\sin(\omega_0 t) + \cos(\omega_0 t)\right]\Big|_0^{T_0/2} \\ &= \frac{A\left(1+e^{-jn\pi}\right)}{\omega_0 T_0 (1-n^2)},\ n \neq \pm 1 \end{aligned}$$

For $n = 1$, the integral is

$$X_1 = \frac{A}{T_0}\int_0^{T_0/2} \sin(\omega_0 t)\left[\cos(jn\omega_0 t) - j\sin(jn\omega_0 t)\right] dt = -\frac{jA}{4} = -X_{-1}^*$$

This is the same result as given in Table 2.1.

Problem 2.19

(a) Use Parseval's theorem to get

$$P_{|nf_0| \le 1/\tau} = \sum_{n=-N}^{N} |X_n|^2 = \sum_{n=-N}^{N} \left(\frac{A\tau}{T_0}\right)^2 \operatorname{sinc}^2(nf_0\tau)$$

where N is an appropriately chosen limit on the sum. We are given that only frequences for which $|nf_0| \le 1/\tau$ are to be included. This is the same as requiring that $|n| \le 1/\tau f_0 = T_0/\tau = 2$. Also, for a pulse train, $P_{\text{total}} = A^2\tau/T_0$ and, in this case, $P_{\text{total}} = A^2/2$. Thus

$$\begin{aligned} \frac{P_{|nf_0| \le 1/\tau}}{P_{\text{total}}} &= \frac{2}{A^2}\sum_{n=-2}^{2}\left(\frac{A}{2}\right)^2 \operatorname{sinc}^2(nf_0\tau) \\ &= \frac{1}{2}\sum_{n=-2}^{2} \operatorname{sinc}^2(nf_0\tau) \\ &= \frac{1}{2}\left[1 + 2\left(\operatorname{sinc}^2(1/2) + \operatorname{sinc}^2(1)\right)\right] \\ &= \frac{1}{2}\left[1 + 2\left(\frac{2}{\pi}\right)^2\right] = 0.91 \end{aligned}$$

(b) In this case, $|n| \leq 5$, $P_{\text{total}} = A^2/5$, and

$$\begin{aligned}
\frac{P_{|nf_0| \leq 1/\tau}}{P_{\text{total}}} &= \frac{1}{5}\sum_{n=-5}^{5} \text{sinc}^2\left(n/5\right) \\
&= \frac{1}{5}\left\{1 + 2\left[(0.9355)^2 + (0.7568)^2 + (0.5046)^2 + (0.2339)^2\right]\right\} \\
&= 0.90
\end{aligned}$$

Problem 2.20

(a) The integral for Y_n is

$$Y_n = \frac{1}{T_0}\int_{T_0} y\left(t\right) e^{-jn\omega_0 t} dt = \frac{1}{T_0}\int_0^{T_0} x\left(t - t_0\right) e^{-jn\omega_0 t} dt$$

Let $t' = t - t_0$, which results in

$$Y_n = \left[\frac{1}{T_0}\int_{-t_0}^{T_0 - t_0} x\left(t'\right) e^{-jn\omega_0 t'} dt'\right] e^{-jn\omega_0 t_0} = X_n e^{-jn\omega_0 t_0}$$

(b) Note that

$$y\left(t\right) = A\cos\omega_0 t = A\sin\left(\omega_0 t + \pi/2\right) = A\sin\left[\omega_0\left(t + \pi/2\omega_0\right)\right]$$

Thus, t_0 in the theorem proved in part (a) here is $-\pi/2\omega_0$. By Euler's theorem, a sine wave can be expressed as

$$\sin\left(\omega_0 t\right) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

Its Fourier coefficients are therefore $X_1 = \frac{1}{2j}$ and $X_{-1} = -\frac{1}{2j}$. According to the theorem proved in part (a), we multiply these by the factor

$$e^{-jn\omega_0 t_0} = e^{-jn\omega_0(-\pi/2\omega_0)} = e^{jn\pi/2}$$

For $n = 1$, we obtain

$$Y_1 = \frac{1}{2j}e^{j\pi/2} = \frac{1}{2}$$

For $n = -1$, we obtain

$$Y_{-1} = -\frac{1}{2j}e^{-j\pi/2} = \frac{1}{2}$$

which gives the Fourier series representation of a cosine wave as

$$y\left(t\right) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} = \cos\omega_0 t$$

We could have written down this Fourier representation directly by using Euler's theorem.

Problem 2.21
(a) Use the Fourier series of a triangular wave as given in Table 2.1 with $A = 1$ and $t = 0$ to obtain the series

$$1 = \cdots + \frac{4}{25\pi^2} + \frac{4}{9\pi^2} + \frac{4}{\pi^2} + \frac{4}{\pi^2} + \frac{4}{9\pi^2} + \frac{4}{25\pi^2} + \cdots$$

Multiply both sides by $\frac{\pi^2}{8}$ to get the series in given in the problem. Therefore, its sum is $\frac{\pi^2}{8}$.
(b) Use the Fourier series of a square wave (specialize the Fourier series of a pulse train) with $A = 1$ and $t = 0$ to obtain the series

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

Multiply both sides by $\frac{\pi}{4}$ to get the series in the problem statement. Hence, the sum is $\frac{\pi}{4}$.

Problem 2.22
(a) In the expression for the Fourier series of a pulse train (Table 2.1), let $t_0 = -T_0/8$ and $\tau = T_0/4$ to get

$$X_n = \frac{A}{4}\text{sinc}\left(\frac{n}{4}\right)\exp\left(j\frac{\pi n f_0}{4}\right)$$

(b) The amplitude spectrum is the same as for part (a) except that $X_0 = \frac{3A}{4}$. Note that this can be viewed as having a sinc-function envelope with zeros at multiples of $\frac{4}{3T_0}$. The phase spectrum can be obtained from that of part (a) by adding a phase shift of π for negative frequencies and subtracting π for postitive frequencies (or vice versa).

Problem 2.23
(a) There is no line at dc; otherwise it looks like a squarewave spectrum.
(b) Note that

$$x_A(t) = K\frac{dx_B(t)}{dt}$$

where K is a suitably chosen constant. The relationship between spectral components is therefore

$$X_n^A = K(jn\omega_0)X_n^B$$

where the superscript A refers to $x_A(t)$ and B refers to $x_B(t)$.

Problem 2.24
(a) This is the right half of a triangle waveform of width τ and height A, or $A(1-t/\tau)$. Therefore, the Fourier transform is

$$\begin{aligned} X_1(f) &= A\int_0^{\tau}(1-t/\tau)\,e^{-j2\pi ft}dt \\ &= \frac{A}{j2\pi f}\left[1-\frac{1}{j2\pi f\tau}\left(1-e^{-j2\pi f\tau}\right)\right] \end{aligned}$$

where a table of integrals has been used.
(b) Since $x_2(t) = x_1(-t)$ we have, by the time reversal theorem, that

$$\begin{aligned} X_2(f) &= X_1^*(f) = X_1(-f) \\ &= \frac{A}{-j2\pi f}\left[1+\frac{1}{j2\pi f\tau}\left(1-e^{j2\pi f\tau}\right)\right] \end{aligned}$$

(c) Since $x_3(t) = x_1(t) - x_2(t)$ we have, after some simplification, that

$$\begin{aligned} X_3(f) &= X_1(f) - X_2(f) \\ &= \frac{jA}{\pi f}\mathrm{sinc}(2f\tau) \end{aligned}$$

(d) Since $x_4(t) = x_1(t) + x_2(t)$ we have, after some simplification, that

$$\begin{aligned} X_4(f) &= X_1(f) + X_2(f) \\ &= A\tau\frac{\sin^2(\pi f\tau)}{(\pi f\tau)^2} \\ &= A\tau\mathrm{sinc}^2(f\tau) \end{aligned}$$

This is the expected result, since $x_4(t)$ is really a triangle function.

Problem 2.25
(a) Using a table of Fourier transforms and the time reversal theorem, the Fourierr transform of the given signal is

$$X(f) = \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}$$

Note that $x(t) \to \mathrm{sgn}(t)$ in the limit as $\alpha \to 0$. Taking the limit of the above Fourier transform as $\alpha \to 0$, we deduce that

$$F[\mathrm{sgn}(t)] = \frac{1}{j2\pi f} - \frac{1}{-j2\pi f} = \frac{1}{j\pi f}$$

(b) Using the given relationship between the unit step and the signum function and the linearity property of the Fourier transform, we obtain

$$\begin{aligned} F\left[u\left(t\right)\right] &= \frac{1}{2}F\left[\operatorname{sgn}\left(t\right)\right]+\frac{1}{2}F\left[1\right] \\ &= \frac{1}{j2\pi f}+\frac{1}{2}\delta\left(f\right) \end{aligned}$$

(c) The same result as obtained in part (b) is obtained.

Problem 2.26

(a) Two differentiations give

$$\frac{d^2x_1\left(t\right)}{dt^2}=\frac{d\delta\left(t\right)}{dt}-\delta\left(t-2\right)+\delta\left(t-3\right)$$

Application of the differentiation theorem of Fourierr transforms gives

$$\left(j2\pi f\right)^2X_1\left(f\right)=\left(j2\pi f\right)\left(1\right)-1\cdot e^{-j4\pi f}+1\cdot e^{-j6\pi f}$$

where the time delay theorem and the Fourier transform of a unit impulse have been used. Dividing both sides by $\left(j2\pi f\right)^2$, we obtain

$$X_1\left(f\right)=\frac{1}{j2\pi f}-\frac{e^{-j4\pi f}-e^{-j6\pi f}}{\left(j2\pi f\right)^2}=\frac{1}{j2\pi f}-\frac{e^{-j5\pi f}}{j2\pi f}\operatorname{sinc}\left(2f\right)$$

(b) Two differentiations give

$$\frac{d^2x_2\left(t\right)}{dt^2}=\delta\left(t\right)-2\delta\left(t-1\right)+\delta\left(t-2\right)$$

Application of the differentiation theorem gives

$$\left(j2\pi f\right)^2X_2\left(f\right)=1-2e^{-j2\pi f}+e^{-j4\pi f}$$

Dividing both sides by $\left(j2\pi f\right)^2$, we obtain

$$X_2\left(f\right)=\frac{1-2e^{-j2\pi f}+e^{-j4\pi f}}{\left(j2\pi f\right)^2}=\operatorname{sinc}^2\left(f\right)e^{-j2\pi f}$$

(c) Two differentiations give

$$\frac{d^2x_3\left(t\right)}{dt^2}=\delta\left(t\right)-\delta\left(t-1\right)-\delta\left(t-2\right)+\delta\left(t-3\right)$$

Application of the differentiation theorem gives

$$(j2\pi f)^2 X_3(f) = 1 - e^{-j2\pi f} - e^{-j4\pi f} + e^{-j6\pi f}$$

Dividing both sides by $(j2\pi f)^2$, we obtain

$$X_3(f) = \frac{1 - e^{-j2\pi f} - e^{-j4\pi f} + e^{-j6\pi f}}{(j2\pi f)^2}$$

(d) Two differentiations give

$$\frac{d^2 x_4(t)}{dt^2} = 2\Pi(t - 1/2) - 2\delta(t-1) - 2\frac{d\delta(t-2)}{dt}$$

Application of the differentiation theorem gives

$$(j2\pi f)^2 X_4(f) = 2\text{sinc}(f)\, e^{-j\pi f} - 2e^{-j2\pi f} - 2(j2\pi f)\, e^{-j4\pi f}$$

Dividing both sides by $(j2\pi f)^3$, we obtain

$$X_4(f) = \frac{2e^{-j2\pi f} + (j2\pi f)\, e^{-j4\pi f} - \text{sinc}(f)\, e^{-j\pi f}}{2(\pi f)^2}$$

Problem 2.27
(a) This is an odd signal, so its Fourier transform is odd and purely imaginary.
(b) This is an even signal, so its Fourier transform is even and purely real.
(c) This is an odd signal, so its Fourier transform is odd and purely imaginary.
(d) This signal is neither even nor odd signal, so its Fourier transform is complex.
(e) This is an even signal, so its Fourier transform is even and purely real.
(f) This signal is even, so its Fourier transform is real and even.

Problem 2.28
(a) Using superposition, time delay, and the Fourier transform of an impulse, we obtain

$$X_1(f) = e^{j16\pi t} + 2 + e^{-j16\pi t} = 4\cos^2(6\pi t)$$

The Fourier transform is even and real because the signal is even.
(b) Using superposition, time delay, and the Fourierr transform of an impulse, we obtain

$$X_2(f) = e^{j12\pi t} - e^{-j12\pi t} = 2j\sin(12\pi f)$$

The Fourier transform is odd and imaginary because the signal is odd.

(c) The Fourier transform is

$$X_3(f) = \sum_{n=0}^{4} \left(n^2 + 1\right) e^{-j4\pi nf}$$

It is complex because the signal is neither even nor odd.

Problem 2.29

(a) The Fourier transform of this signal is

$$X_1(f) = \frac{2(1/3)}{1 + (2\pi f/3)^2} = \frac{2/3}{1 + [f/(3/2\pi)]^2}$$

Thus, the energy spectral density is

$$G_1(f) = \left\{ \frac{2/3}{1 + [f/(3/2\pi)]^2} \right\}^2$$

(b) The Fourier transform of this signal is

$$X_2(f) = \frac{2}{3}\Pi\left(\frac{f}{30}\right)$$

Thus, the energy spectral density is

$$X_2(f) = \frac{4}{9}\Pi^2\left(\frac{f}{30}\right) = \frac{4}{9}\Pi\left(\frac{f}{30}\right)$$

(c) The Fourier transform of this signal is

$$X_3(f) = \frac{4}{5}\mathrm{sinc}\left(\frac{f}{5}\right)$$

so the energy spectral density is

$$G_3(f) = \frac{16}{25}\mathrm{sinc}^2\left(\frac{f}{5}\right)$$

(d) The Fourier transform of this signal is

$$X_4(f) = \frac{2}{5}\left[\mathrm{sinc}\left(\frac{f-20}{5}\right) + \mathrm{sinc}\left(\frac{f+20}{5}\right)\right]$$

so the energy spectral density is

$$G_4(f) = \frac{4}{25}\left[\operatorname{sinc}\left(\frac{f-20}{5}\right) + \operatorname{sinc}\left(\frac{f+20}{5}\right)\right]^2$$

Problem 2.30

(a) Use the transform pair

$$x_1(t) = e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{\alpha + j2\pi f}$$

Using Rayleigh's energy theorem, we obtain the integral relationship

$$\int_{-\infty}^{\infty} |X_1(f)|^2\, df = \int_{-\infty}^{\infty} \frac{df}{\alpha^2 + (2\pi f)^2} df = \int_{-\infty}^{\infty} |x_1(t)|^2\, dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}$$

(b) Use the transform pair

$$x_2(t) = \frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow \operatorname{sinc}(\tau f) = X_2(f)$$

Rayleigh's energy theorem gives

$$\begin{aligned}\int_{-\infty}^{\infty} |X_2(f)|^2\, df &= \int_{-\infty}^{\infty} \operatorname{sinc}^2(\tau f)\, df = \int_{-\infty}^{\infty} |x_2(t)|^2\, dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\tau^2}\Pi^2\left(\frac{t}{\tau}\right) dt = \int_{-\tau/2}^{\tau/2} \frac{dt}{\tau^2} = \frac{1}{\tau}\end{aligned}$$

(c) Use the transform pair

$$x_3(t) = e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

The desired integral, by Rayleigh's energy theorem, is

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} |X_3(f)|^2\, df = \int_{-\infty}^{\infty} \left[\frac{1}{\alpha^2 + (2\pi f)^2}\right]^2 df \\ &= \frac{1}{(2\alpha)^2}\int_{-\infty}^{\infty} e^{-2\alpha|t|} dt = \frac{1}{2\alpha^2}\int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{4\alpha^3}\end{aligned}$$

(d) Use the transform pair

$$\frac{1}{\tau}\Lambda\left(\frac{t}{\tau}\right) \longleftrightarrow \operatorname{sinc}^2(\tau f)$$

The desired integral, by Rayleigh's energy theorem, is

$$\begin{aligned} I_4 &= \int_{-\infty}^{\infty} |X_4(f)|^2 df = \int_{-\infty}^{\infty} \operatorname{sinc}^4(\tau f) df \\ &= \frac{1}{\tau^2}\int_{-\infty}^{\infty} \Lambda^2(t/\tau)\, dt = \frac{2}{\tau^2}\int_0^{\tau} [1-(t/\tau)]^2 dt \\ &= \frac{2}{\tau}\int_0^1 [1-u]^2 du = \frac{2}{3\tau} \end{aligned}$$

Problem 2.31

(a) The convolution operation gives

$$y_1(t) = \begin{cases} 0,\ t \le \tau - 1/2 \\ \frac{1}{\alpha}\left[1 - e^{-\alpha(t-\tau+1/2)}\right],\ \tau - 1/2 < t \le \tau + 1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-\tau-1/2)} - e^{-\alpha(t-\tau+1/2)}\right],\ t > \tau + 1/2 \end{cases}$$

(b) The convolution of these two signals gives

$$y_2(t) = \Lambda(t) + \operatorname{tr}(t)$$

where $\operatorname{tr}(t)$ is a trapezoidal function given by

$$\operatorname{tr}(t) = \begin{cases} 0,\ t < -3/2 \text{ or } t > 3/2 \\ 1,\ -1/2 \le t \le 1/2 \\ 3/2 + t,\ -3/2 \le t < -1/2 \\ 3/2 - t,\ 1/2 \le t < 3/2 \end{cases}$$

(c) The convolution results in

$$y_3(t) = \int_{-\infty}^{\infty} e^{-\alpha|\lambda|}\Pi(\lambda - t)\, d\lambda = \int_{t-1/2}^{t+1/2} e^{-\alpha|\lambda|} d\lambda$$

Sketches of the integrand for various values of t gives the following cases:

$$y_3(t) = \begin{cases} \int_{t-1/2}^{t+1/2} e^{\alpha\lambda} d\lambda,\ t \le -1/2 \\ \int_{t-1/2}^{0} e^{\alpha\lambda} d\lambda + \int_0^{t+1/2} e^{-\alpha\lambda} d\lambda,\ -1/2 < t \le 1/2 \\ \int_{t-1/2}^{t+1/2} e^{-\alpha\lambda} d\lambda,\ t > 1/2 \end{cases}$$

Integration of these three cases gives

$$y_3(t) = \begin{cases} \frac{1}{\alpha}\left[e^{\alpha(t+1/2)} - e^{\alpha(t-1/2)}\right],\ t \le -1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-1/2)} - e^{-\alpha(t+1/2)}\right],\ -1/2 < t \le 1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-1/2)} - e^{-\alpha(t+1/2)}\right],\ t > 1/2 \end{cases}$$

(d) The convolution gives

$$y_4(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

Problem 2.32

(a) Using the convolution and time delay theorems, we obtain

$$\begin{aligned} Y_1(f) &= F\left[e^{-\alpha t}u(t) * \Pi(t-\tau)\right] \\ &= F\left[e^{-\alpha t}u(t)\right] F\left[\Pi(t-\tau)\right] \\ &= \frac{1}{\alpha + j2\pi f}\mathrm{sinc}(f)\, e^{-j2\pi f\tau} \end{aligned}$$

(b) The superposition and convolution theorems give

$$\begin{aligned} Y_2(f) &= F\left\{\left[\Pi(t/2) + \Pi(t)\right] * \Pi(t)\right\} \\ &= \left[2\mathrm{sinc}(2f) + \mathrm{sinc}(f)\right]\mathrm{sinc}(f) \end{aligned}$$

(c) By the convolution theorem

$$\begin{aligned} Y_3(f) &= F\left[e^{-\alpha|t|} * \Pi(t)\right] \\ &= \frac{2\alpha}{\alpha^2 + (2\pi f)^2}\mathrm{sinc}(f) \end{aligned}$$

(d) By the convolution theorem (note, also, that the integration theorem can be applied directly)

$$\begin{aligned} Y_4(f) &= F\left[x(t) * u(t)\right] \\ &= X(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ &= \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\,\delta(f) \end{aligned}$$

Problem 2.33

(a) The normalized inband energy is

$$\frac{E_1(|f| \le W)}{E_{\text{total}}} = \frac{2}{\pi}\tan^{-1}\left(\frac{2\pi W}{\alpha}\right)$$

(b) The result is

$$\frac{E_1\left(|f| \leq W\right)}{E_{\text{total}}} = 2\int_0^{\tau W} \text{sinc}^2\left(u\right) du$$

The integration must be carried out numerically.

Problem 2.34
(a) By the modulation theorem

$$\begin{aligned} X\left(f\right) &= \frac{AT_0}{4}\left\{\text{sinc}\left[\left(f-f_0\right)\frac{T_0}{2}\right] + \text{sinc}\left[\left(f+f_0\right)\frac{T_0}{2}\right]\right\} \\ &= \frac{AT_0}{4}\left\{\text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}-1\right)\right] + \text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}+1\right)\right]\right\} \end{aligned}$$

(b) Use the superposition and modulation theorems to get

$$X\left(f\right) = \frac{AT_0}{4}\left\{\text{sinc}\left[\frac{f}{2f_0}\right] + \frac{1}{2}\left[\text{sinc}\frac{1}{2}\left(\frac{f}{f_0}-2\right) + \text{sinc}\frac{1}{2}\left(\frac{f}{f_0}+2\right)\right]\right\}$$

Problem 2.35
Combine the exponents of the two factors in the integrand of the Fourier transform integral, complete the square, and use the given definite integral.

Problem 2.36
Consider the development below:

$$x\left(t\right) * x\left(-t\right) = \int_{-\infty}^{\infty} x\left(-\lambda\right) x\left(t-\lambda\right) d\lambda = \int_{-\infty}^{\infty} x\left(\beta\right) x\left(t+\beta\right) d\beta$$

where $\beta = -\lambda$ has been substituted. Rename variables to obtain

$$R\left(\tau\right) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x\left(\beta\right) x\left(t+\beta\right) d\beta$$

Problem 2.37
The result is an even triangular wave with zero average value of period T_0. It makes no difference whether the original square wave is even or odd or neither.

Problem 2.38
Fourier transform both sides of the differential equation using the differentiation theorem of Fourier transforms to get

$$[j2\pi f + a]\, Y(f) = [j2\pi bf + c]\, X(f)$$

Therefore, the transfer function is

$$H(f) = \frac{Y(f)}{X(f)} = \frac{c + j2\pi bf}{a + j2\pi f}$$

The amplitude response function is

$$|H(f)| = \frac{\sqrt{c^2 + (2\pi bf)^2}}{\sqrt{a^2 + (2\pi f)^2}}$$

and the phase response is

$$\arg[H(f)] = \tan^{-1}\left(\frac{2\pi bf}{c}\right) - \tan^{-1}\left(\frac{2\pi f}{a}\right)$$

Amplitude and phase responses for various values of the constants are plotted below.

Problem 2.39
(a) The find the unit impulse response, write $H(f)$ as

$$H(f) = 1 - \frac{5}{5 + j2\pi f}$$

Inverse Fourier transforming gives

$$h(t) = \delta(t) - 5e^{-5t}u(t)$$

(b) Use the transform pair

$$Ae^{-\alpha t}u(t) \longleftrightarrow \frac{A}{\alpha + j2\pi f}$$

and the time delay theorem to find the unit impulse as

$$h(t) = \frac{2}{5}e^{-\frac{8}{15}(t-3)}u(t-3)$$

Problem 2.40
Use the transform pair for a sinc function to find that

$$Y(f) = \Pi\left(\frac{f}{2B}\right)\Pi\left(\frac{f}{2W}\right)$$

(a) If $W < B$, it follows that

$$Y(f) = \Pi\left(\frac{f}{2W}\right)$$

because $\Pi\left(\frac{f}{2B}\right) = 1$ throughout the region where $\Pi\left(\frac{f}{2W}\right)$ is nonzero.
(b) If $W > B$, it follows that

$$Y(f) = \Pi\left(\frac{f}{2B}\right)$$

because $\Pi\left(\frac{f}{2W}\right) = 1$ throughout the region where $\Pi\left(\frac{f}{2B}\right)$ is nonzero.

Problem 2.41
(a) Replace the capacitors with $1/j\omega C$ which is their ac-equivalent impedance. Call the junction of the input resistor, feedback resistor, and capacitors 1. Call the junction at the positive input of the operational amplifier 2. Call the junction at the negative input of the operational amplifier 3. Write down the KCL equations at these three junctions. Use the constraint equation for the operational amplifier, which is $V_2 = V_3$, and the definitions for ω_0, Q, and K to get the given transfer function.
(d) Combinations of components giving

$$RC = 2.3 \times 10^{-4} \text{ seconds}$$

and

$$\frac{R_a}{R_b} = 2.5757$$

will work.

Problem 2.42
(a) By long division

$$H(f) = 1 - \frac{R_1/L}{\frac{R_1+R_2}{L} + j2\pi f}$$

Using the transforms of a delta function and a one-sided exponential, we obtain

$$h(t) = \delta(t) - \frac{R_1}{L}\exp\left(-\frac{R_1+R_2}{L}t\right)u(t)$$

(b) Substituting the ac-equivalent impedance for the inductor and using voltage division, the transfer function is

$$H(f) = \frac{R_2}{R_1+R_2}\frac{j2\pi fL}{\frac{R_1R_2}{R_1+R_2}+j2\pi fL} = \frac{R_2}{R_1+R_2}\left(1-\frac{(R_1 \parallel R_2)/L}{(R_1 \parallel R_2)/L+j2\pi f}\right)$$

Therefore, the impulse response is

$$h(t) = \frac{R_2}{R_1+R_2}\left[\delta(t) - \frac{R_1R_2}{(R_1+R_2)L}\exp\left(-\frac{R_1R_2}{(R_1+R_2)L}t\right)u(t)\right]$$

Problem 2.43

The Payley-Wiener criterion gives the integral

$$I = \int_{-\infty}^{\infty}\frac{\beta f^2}{1+f^2}df$$

which does not converge. Hence, the given function is not suitable as the transfer function of a causal LTI system.

Problem 2.44

(a) The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty}|h_1(t)|\,dt &= \int_{-\infty}^{\infty}|\exp(-\alpha t)\cos(2\pi f_0 t)\,u(t)|\,dt \\ &= \int_0^{\infty}\exp(-\alpha t)\,|\cos(2\pi f_0 t)|\,dt < \int_0^{\infty}\exp(-\alpha t)\,dt = \frac{1}{\alpha} < \infty\end{aligned}$$

which follows because $|\cos(2\pi f_0 t)| \leq 1$.

(b) The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty}|h_2(t)|\,dt &= \int_{-\infty}^{\infty}|\cos(2\pi f_0 t)\,u(t)|\,dt \\ &= \int_0^{\infty}|\cos(2\pi f_0 t)|\,dt \to \infty\end{aligned}$$

which follows by integrating one period of $|\cos(2\pi f_0 t)|$ and noting that the total integral is the limit of one period of area times N as $N \to \infty$.

(c) The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty}|h_3(t)|\,dt &= \int_{-\infty}^{\infty}\frac{1}{t}u(t-1)\,dt \\ &= \int_1^{\infty}\frac{dt}{t} = \ln(t)|_1^{\infty} \to \infty\end{aligned}$$

Problem 2.45
The energy spectral density of the output is

$$G_y(f) = |H(f)|^2 |X(f)|^2$$

where

$$X(f) = \frac{1}{2 + j2\pi f}$$

Hence

$$G_y(f) = \frac{100}{\left[9 + (2\pi f)^2\right]\left[4 + (2\pi f)^2\right]}$$

Problem 2.46
Using the Fourier coefficients of a half-rectified sine wave from Table 2.1 and noting that those of a half-rectified cosine wave are related by

$$X_{c_n} = X_{s_n} e^{-jn\pi/2}$$

The fundamental frequency is 10 Hz. The ideal rectangular filter passes all frequencies less than 13 Hz and rejects all frequencies greater than 13 Hz. Therefore

$$y(t) = \frac{3A}{\pi} - \frac{3A}{2}\cos(20\pi t)$$

Problem 2.47

(a) The 90% energy containment bandwidth is given by

$$B_{90} = \frac{\alpha}{2\pi}\tan(0.45\pi) = 1.0055\alpha$$

(b) For this case, using $X_2(f) = \Pi(f/2W)$, we obtain

$$B_{90} = 0.9W$$

(c) Numerical integration gives

$$B_{90} = 0.85/\tau$$

Problem 2.48
From Example 2.7

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \cdots\right]$$

From the transfer function of a Hilbert transformer, we find its output in response to the above input to be

$$y(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\cos\left(\omega_0 t - \frac{\pi}{2}\right) - \frac{1}{3}\cos\left(3\omega_0 t - \frac{\pi}{2}\right) + \cdots\right]$$

Problem 2.49
(a) Amplitude distortion; no phase distortion.
(b) No amplitude distortion; phase distortion.
(c) No amplitude distortion; no phase distortion.
(d) No amplitude distortion; no phase distortion.

Problem 2.50
The transfer function corresponding to this impulse response is

$$H(f) = \frac{2}{3 + j2\pi f} = \frac{2}{\sqrt{9 + (2\pi f)^2}} \exp\left[-j\tan\left(\frac{2\pi f}{3}\right)\right]$$

The group delay is

$$\begin{aligned} T_g(f) &= -\frac{1}{2\pi}\frac{d}{df}\left[-\tan\left(\frac{2\pi f}{3}\right)\right] \\ &= \frac{3}{9 + (2\pi f)^2} \end{aligned}$$

The phase delay is

$$T_p(f) = -\frac{\theta(f)}{2\pi f} = \frac{\tan\left(\frac{2\pi f}{3}\right)}{2\pi f}$$

Problem 2.51
The group and phase delays are, respectively,

$$\begin{aligned} T_g(f) &= \frac{0.1}{1 + (0.2\pi f)^2} - \frac{0.333}{1 + (0.667\pi f)^2} \\ T_p(f) &= \frac{1}{2\pi f}\left[\tan(0.2\pi f) - \tan(0.667\pi f)\right] \end{aligned}$$

Problem 2.52
In terms of the input spectrum, the output spectrum is

$$\begin{aligned}
Y(f) &= X(f) + 0.2X(f) * X(f) \\
&= 4\left[\Pi\left(\frac{f-20}{6}\right) + \Pi\left(\frac{f+20}{6}\right)\right] \\
&\quad +3.2\left[\Pi\left(\frac{f-20}{6}\right) + \Pi\left(\frac{f+20}{6}\right)\right] * \left[\Pi\left(\frac{f-20}{6}\right) + \Pi\left(\frac{f+20}{6}\right)\right] \\
&= 4\left[\Pi\left(\frac{f-20}{6}\right) + \Pi\left(\frac{f+20}{6}\right)\right] \\
&\quad +3.2\left[6\Lambda\left(\frac{f-40}{6}\right) + 12\Lambda\left(\frac{f}{6}\right) + 6\Lambda\left(\frac{f+40}{6}\right)\right]
\end{aligned}$$

where $\Lambda(t)$ is an isoceles triangle of unit height going from -1 to 1. The student should sketch the output spectrum given the above analytical result..

Problem 2.53
(a) The output of the nonlinear device is

$$y(t) = 1.075\cos(2000\pi t) + 0.025\cos(6000\pi t)$$

The steadtstate filter output in response to this input is

$$z(t) = 1.075\,|H(1000)|\cos(2000\pi t) + 0.025\,|H(3000)|\cos(6000\pi t)$$

so that the THD is

$$\begin{aligned}
\text{THD} &= \frac{(1.075)^2\,|H(1000)|^2}{(0.025)^2\,|H(3000)|^2} \\
&= \frac{43^2}{1+4Q^2(1000-3000)^2} \\
&= \frac{1849}{1+16\times 10^6 Q^2}
\end{aligned}$$

where $H(f)$ is the transfer function of the filter.
(b) For THD $= 0.005\% = 0.00005$, the equation for Q becomes

$$\frac{1849}{1+16\times 10^6 Q^2} = 0.00005$$

or

$$\begin{aligned} 6 \times 10^6 Q^2 &= 1849 \times 2 \times 10^4 - 1 \\ Q &= 1.52 \end{aligned}$$

Problem 2.54
Frequency components are present in the output at radian frequencies of 0, ω_1, ω_2, $2\omega_1$, $2\omega_2$, $\omega_1 - \omega_2$, $\omega_1 + \omega_2$,$3\omega_1$, $3\omega_2$, $2\omega_2 - \omega_1$, $\omega_1 + 2\omega_2$, $2\omega_1 - \omega_2$, $2\omega_1 + \omega_2$. To use this as a frequency multiplier, pass the components at $2\omega_1$ or $2\omega_2$ to use as a doubler, or at $3\omega_1$ or $3\omega_2$ to use as a tripler.

Problem 2.55
Write the transfer function as

$$H(f) = H_0 e^{-j2\pi f t_0} - H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$$

Use the inverse Fourier transform of a constant, the delay theorem, and the inverse Fourier transform of a rectangular pulse function to get

$$h(t) = H_0 \delta(t - t_0) - 2BH_0 \text{sinc}\left[2B(t - t_0)\right]$$

Problem 2.56
(a) The Fourier transform of this signal is

$$X(f) = A\sqrt{2\pi b^2} \exp\left(-2\pi^2 \tau^2 f^2\right)$$

By definition, using a table of integrals,

$$T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)|\, dt = \sqrt{2\pi}\tau$$

Similarly,

$$W = \frac{1}{2X(0)} \int_{-\infty}^{\infty} |X(f)|\, df = \frac{1}{2\sqrt{2\pi}\tau}$$

Therefore,

$$2WT = \frac{2}{2\sqrt{2\pi}\tau}\sqrt{2\pi}\tau = 1$$

(b) The Fourier transform of this signal is

$$X(f) = \frac{2A/\alpha}{1 + (2\pi f/\alpha)^2}$$

The pulse duration is

$$T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)|\, dt = \frac{2}{\alpha}$$

The bandwidth is

$$W = \frac{1}{2X(0)} \int_{-\infty}^{\infty} |X(f)|\, df = \frac{\alpha}{4}$$

Thus,

$$2WT = 2\left(\frac{\alpha}{4}\right)\left(\frac{2}{\alpha}\right) = 1$$

Problem 2.57

(a) The poles for a second order Butterworth filter are given by

$$s_1 = s_2^* = -\frac{\omega_3}{\sqrt{2}}(1 - j)$$

where ω_3 is the 3-dB cutoff frequency of the Butterworth filter. Its s-domain transfer function is

$$H(s) = \frac{\omega_3^2}{\left[s + \frac{\omega_3}{\sqrt{2}}(1-j)\right]\left[s + \frac{\omega_3}{\sqrt{2}}(1+j)\right]} = \frac{\omega_3^2}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2}$$

Letting $\omega_3 = 2\pi f_3$ and $s = j\omega = j2\pi f$, we obtain

$$H(j2\pi f) = \frac{4\pi^2 f_3^2}{-4\pi^2 f^2 + \sqrt{2}(2\pi f_3)(j2\pi f) + 4\pi^2 f_3^2} = \frac{f_3^2}{-f^2 + j\sqrt{2} f_3 f + f_3^2}$$

(b) If the phase response function of the filter is $\theta(f)$, the group delay is

$$T_g(f) = -\frac{1}{2\pi}\frac{d}{df}[\theta(f)]$$

For the second-order Butterworth filter considered here,

$$\theta(f) = -\tan^{-1}\left(\frac{\sqrt{2} f_3 f}{f_3^2 - f^2}\right)$$

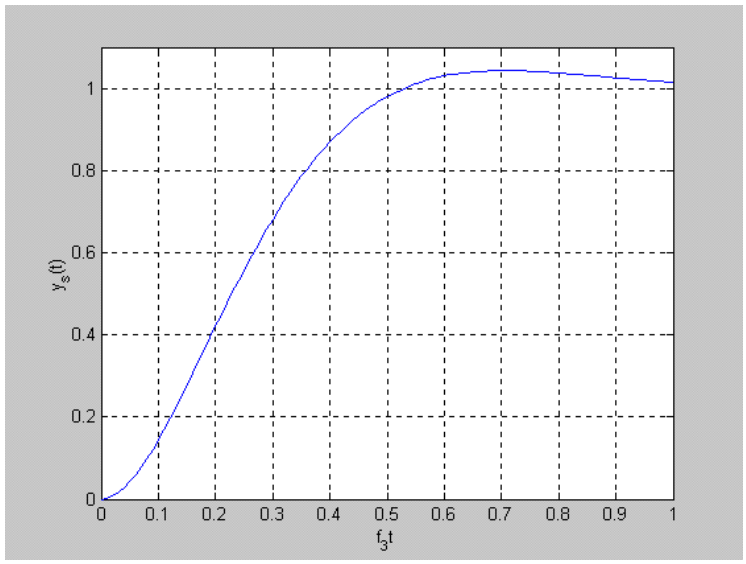

Figure 2.3:

Therefore, the group delay is

$$\begin{aligned} T_g(f) &= \frac{1}{2\pi}\frac{d}{df}\left[\tan^{-1}\left(\frac{\sqrt{2}f_3 f}{f_3^2 - f^2}\right)\right] \\ &= \frac{f_3}{\sqrt{2}\pi}\frac{f_3^2 + f^2}{f_3^4 + f^4} = \frac{1}{\sqrt{2}\pi f_3}\frac{1 + (f/f_3)^2}{1 + (f/f_3)^4} \end{aligned}$$

(c) Use partial fraction expansion of $H(s)$ and then inverse Laplace transform $H(s)/s$ to get the given step response. Plot it and estimate the 10% and 90% times from the plot. From the MATLAB plot of Fig. 2.3, $f_3 t_{10\%} \approx 0.08$ and $f_3 t_{90\%} \approx 0.42$ so that the 10-90 % rise time is about $0.34/f_3$ seconds.

Problem 2.58
(a) 0.5 seconds; (b) and (c) - use sketches to show.

Problem 2.59
(a) The leading edges of the flat-top samples follow the waveform at the sampling instants.
(b) The spectrum is

$$Y(f) = X_\delta(f) H(f)$$

where

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

and

$$H(f) = \tau \mathrm{sinc}(f\tau) \exp(-j\pi f\tau)$$

The latter represents the frequency response of a filter whose impulse response is a square pulse of width τ and implements flat top sampling. If W is the bandwidth of $X(f)$, very little distortion will result if $\tau^{-1} >> W$.

Problem 2.60
(a) The sampling frequency should be large compared with the bandwidth of the signal.
(b) The output spectrum of the zero-order hold circuit is

$$Y(f) = \mathrm{sinc}(T_s f) \sum_{n=-\infty}^{\infty} X(f - nf_s) \exp(-j\pi f T_s)$$

where $f_s = T_s^{-1}$. For small distortion, we want $T_s << W^{-1}$.

Problem 2.61
The lowpass recovery filter can cut off in the range 1.9^+ kHz to 2.1^- kHz.

Problem 2.62
For bandpass sampling and recovery, all but (b) and (e) will work theoretically, although an ideal filter with bandwidth exactly equal to the unsampled signal bandwidth is necessary. For lowpass sampling and recovery, only (f) will work.

Problem 2.63
The Fourier transform is

$$\begin{aligned} Y(f) &= \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0) \\ &\quad + [-j\mathrm{sgn}(f) X(f)] * \left[\frac{1}{2}\delta(f - f_0) e^{-j\pi/2} + \frac{1}{2}\delta(f + f_0) e^{j\pi/2}\right] \\ &= \frac{1}{2}X(f - f_0)[1 - \mathrm{sgn}(f - f_0)] + \frac{1}{2}X(f + f_0)[1 + \mathrm{sgn}(f + f_0)] \end{aligned}$$

Problem 2.64

(a) $\widehat{x}_a(t) = \cos(\omega_0 t - \pi/2) = \sin(\omega_0 t)$, so

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,\widehat{x}(t)\,dt &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 t)\cos(\omega_0 t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{1}{2}\sin(2\omega_0 t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\left.\frac{\cos(2\omega_0 t)}{4\omega_0}\right|_{-T}^{T} = 0
\end{aligned}$$

(b) Use trigonometric identities to express $x(t)$ in terms of sines and cosines. Then find the Hilbert transform of $x(t)$ by phase shifting by $-\pi/2$. Multiply $x(t)$ and $\widehat{x}(t)$ together term by term, use trigonometric identities for the product of sines and cosines, then integrate. The integrand will be a sum of terms similar to that of part (a). The limit as $T \to \infty$ will be zero term-by-term.

(c) Use the integral definition of $\widehat{x}(t)$, take the product, integrate over time to get

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)\,\widehat{x}(t)\,dt &= A^2\int_{-\infty}^{\infty} \Pi(t/\tau)\left[\int_{-\infty}^{\infty} \frac{\Pi(\lambda/\tau)}{\pi(t-\lambda)}d\lambda\right]dt \\
&= A^2\int_{-\tau/2}^{\tau/2}\left[\int_{-\tau/2}^{\tau/2} \frac{1}{\pi(t-\lambda)}d\lambda\right]dt \\
&= A^2\int_{-\tau/2}^{\tau/2} \frac{1}{\pi}\ln\left|\frac{t-\tau/2}{t+\tau/2}\right|dt = 0
\end{aligned}$$

where the result is zero by virtue of the integrand of the last integral being odd.

Problem 2.65

(a) Note that $F[j\widehat{x}(t)] = j[-j\mathrm{sgn}(f)]X(f)$. Hence

$$\begin{aligned}
x_1(t) &= \frac{3}{4}x(t) + \frac{1}{4}j\widehat{x}(t) \to X_1(f) = \frac{3}{4}X(f) + \frac{1}{4}j[-j\mathrm{sgn}(f)]X(f) \\
&= \left[\frac{3}{4} + \frac{1}{4}\mathrm{sgn}(f)\right]X(f) \\
&= \begin{cases} \frac{1}{2}X(f), & f < 0 \\ X(f), & f > 0 \end{cases}
\end{aligned}$$

A sketch is shown in Figure 2.4.

(b) It follows that

$$\begin{aligned} x_2(t) &= \left[\frac{3}{4}x(t) + \frac{3}{4}j\widehat{x}(t)\right]\exp(j2\pi f_0 t) \\ &\rightarrow X_2(f) = \frac{3}{4}\left[1+\operatorname{sgn}(f-f_0)\right]X(f-f_0) \\ &= \begin{cases} 0, & f < f_0 \\ \frac{3}{2}X(f-f_0), & f > f_0 \end{cases} \end{aligned}$$

A sketch is shown in Figure 2.4.
(c) This case has the same spectrum as part (a), except that it is shifted right by W Hz. That is,

$$\begin{aligned} x_3(t) &= \left[\frac{3}{4}x(t) + \frac{1}{4}j\widehat{x}(t)\right]\exp(j2\pi W t) \\ &\rightarrow X_3(f) = \left[\frac{3}{4} + \frac{1}{4}\operatorname{sgn}(f-W)\right]X(f-W) \end{aligned}$$

A sketch is shown in Figure 2.4.
(d) For this signal

$$\begin{aligned} x_4(t) &= \left[\frac{3}{4}x(t) - \frac{1}{4}j\widehat{x}(t)\right]\exp(j\pi W t) \\ &\rightarrow X_4(f) = \left[\frac{3}{4} - \frac{1}{4}\operatorname{sgn}(f-W/2)\right]X(f-W/2) \end{aligned}$$

A sketch is shown in Figure 2.4.

Problem 2.66
(a) The spectrum is

$$X_p(f) = X(f) + j\left[-j\operatorname{sgn}(f)\right]X(f) = \left[1+\operatorname{sgn}(f)\right]X(f)$$

The Fourier transform of $x(t)$ is

$$X(f) = \frac{1}{2}\Pi\left(\frac{f-f_0}{2W}\right) + \frac{1}{2}\Pi\left(\frac{f+f_0}{2W}\right)$$

Thus,

$$X_p(f) = \Pi\left(\frac{f-f_0}{2W}\right) \text{ if } f_0 > 2W$$

(b) The complex envelope is defined by

$$x_p(t) = \tilde{x}(t)\,e^{j2\pi f_0 t}$$

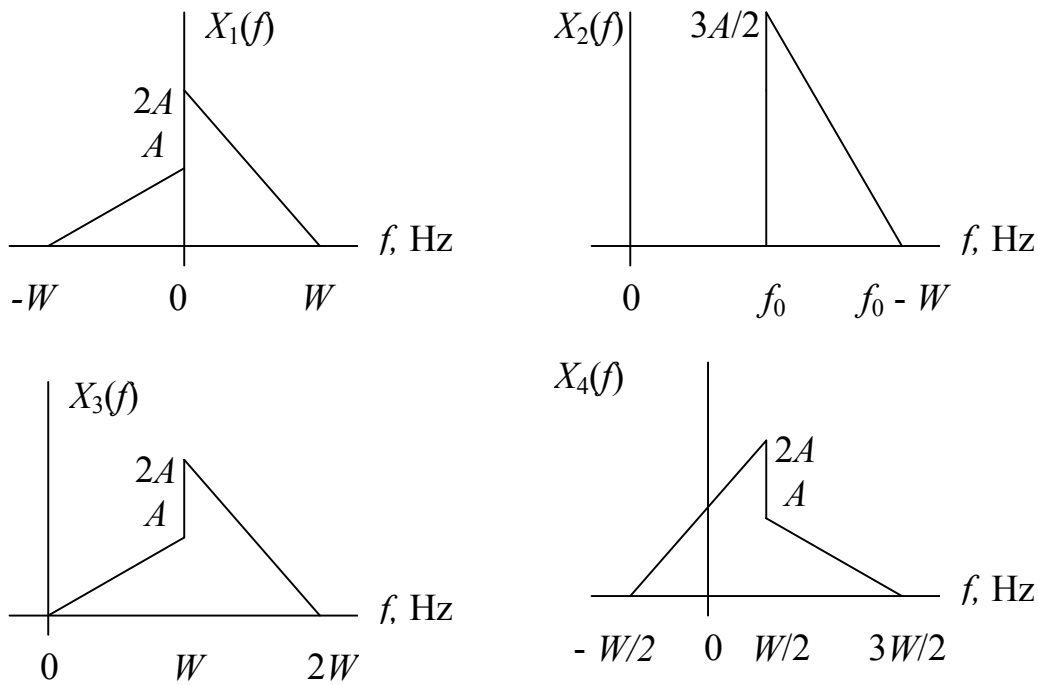
X1(f)
2A
A
f, Hz
-W
0
W
X2(f)
3A/2
f, Hz
0
f0
f0 - W
X3(f)
2A
A
f, Hz
0
W
2W
X4(f)
2A
A
f, Hz
- W/2
0
W/2
3W/2

Figure 2.4:

Therefore

$$\tilde{x}(t) = x_p(t)\, e^{-j2\pi f_0 t}$$

Hence

$$F[\tilde{x}(t)] = F[x_p(t)]|_{f\to f+f_0} = \Pi\left(\frac{f-f_0}{2W}\right)\bigg|_{f\to f+f_0} = \Pi\left(\frac{f}{2W}\right)$$

(c) The complex envelope is

$$\tilde{x}(t) = F^{-1}\left[\Pi\left(\frac{f}{2W}\right)\right] = 2W\ \mathrm{sinc}(2Wt)$$

Problem 2.67

For $t < \tau/2$, the output is zero. For $|t| \le \tau/2$, the result is

$$\begin{aligned} y(t) &= \frac{\alpha/2}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \\ &\quad \times \left\{\cos[2\pi(f_0+\Delta f)t - \theta] - e^{-\alpha(t+\tau/2)}\cos[2\pi(f_0+\Delta f)t + \theta]\right\} \end{aligned}$$

For $t > \tau/2$, the result is

$$\begin{aligned} y(t) &= \frac{(\alpha/2)\, e^{-\alpha t}}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \\ &\quad \times \left\{e^{\alpha\tau/2}\cos[2\pi(f_0+\Delta f)t - \theta] - e^{-\alpha\tau/2}\cos[2\pi(f_0+\Delta f)t + \theta]\right\} \end{aligned}$$

In the above equations, θ is given by

$$\theta = -\tan^{-1}\left(\frac{2\pi\Delta f}{\alpha}\right)$$

2.2 Computer Exercises

Computer Exercise 2.1

```
% ce2_1: Computes generalized Fourier series coefficients for exponentially
% decaying signal, exp(-t)u(t), or sinewave, sin(2*pi*t)
%
tau = input('Enter approximating pulse width: ');
type_wave = input('Enter waveform type: 1 = decaying exponential; 2 = sinewave ');
if type_wave == 1
```

```
    t_max = input('Enter duration of exponential: ');
elseif type_wave == 2
    t_max = 1;
end
clf
a = [];
t = 0:.01:t_max;
cn = tau;
n_max = floor(t_max/tau)
for n = 1:n_max
    LL = (n-1)*tau;
    UL = n*tau;
     if type_wave == 1
        a(n) = (1/cn)*quad(@integrand_exp, LL, UL);
    elseif type_wave == 2
        a(n) = (1/cn)*quad(@integrand_sine, LL, UL);
    end
end
disp(' ')
disp('c_n')
disp(cn)
disp(' ')
disp('Expansion coefficients (approximating pulses not normalized):')
disp(a)
disp(' ')
x_approx = zeros(size(t));
for n = 1:n_max
    x_approx = x_approx + a(n)*phi(t/tau,n-1);
end
if type_wave == 1
    MSE = quad(@integrand_exp2, 0, t_max) - cn*a*a';
elseif type_wave == 2
    MSE = quad(@integrand_sine2, 0, t_max) - cn*a*a';
end
disp('Mean-squared error:')
disp(MSE)
disp(' ')
if type_wave == 1
    plot(t, x_approx), axis([0 t_max -inf inf]), xlabel('t'),
                                ylabel('x(t); x_a_p_p_r_o_x(t)')
```

```
    hold
    plot(t, exp(-t))
     title(['Approximation of exp(-t) over [0, ',num2str(t_max),'] with
                contiguous rectangular pulses of width ',num2str(tau)])
elseif type_wave == 2
    plot(t, x_approx), axis([0 t_max -inf inf]), xlabel('t'),
                        ylabel('x(t); x_a_p_p_r_o_x(t)')
    hold
    plot(t, sin(2*pi*t))
    title(['Approximation of sin(2*pi*t) over [0, ',num2str(t_max),'] with
                    contiguous rectangular pulses of width ',num2str(tau)])
end

% Decaying exponential function for ce_1
%
function z = integrand_exp(t)
z = exp(-t);

% Sine function for ce_1
%
function z = integrand_sine(t)
z = sin(2*pi*t);

% Decaying exponential squared function for ce_1
%
function z = integrand_exp2(t)
z = exp(-2*t);

% Sin^2 function for ce_1
%
function z = integrand_sine2(t)
z = (sin(2*pi*t)).^2;
```

A typical run follows:

```
>> ce2_1
Enter approximating pulse width: 0.125
Enter waveform type: 1 = decaying exponential; 2 = sinewave: 2
n_max =
8
c_n
0.1250
Expansion coefficients (approximating pulses not normalized):
```

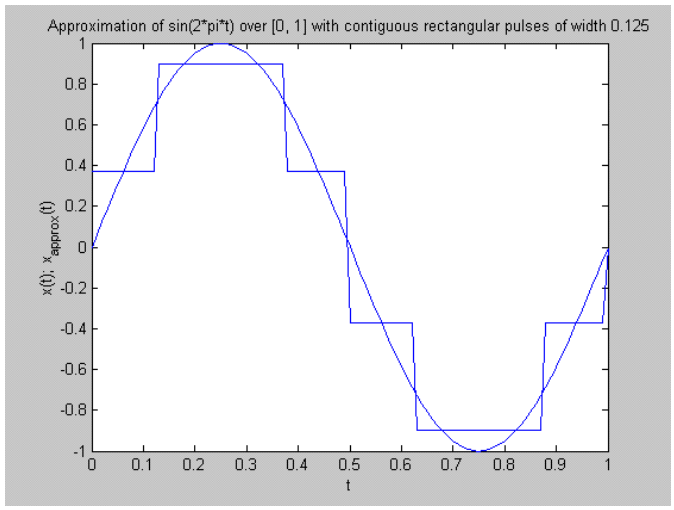

Figure 2.5:

```
Columns 1 through 7
0.3729    0.9003    0.9003    0.3729    -0.3729    -0.9003    -0.9003
Column 8
-0.3729
Mean-squared error:
0.0252
```

Computer Exercise 2.2

```
% ce2_1: Computes generalized Fourier series coefficients for exponentially
% decaying signal, exp(-t)u(t), or sinewave, sin(2*pi*t)
%
tau = input('Enter approximating pulse width: ');
type_wave = input('Enter waveform type: 1 = decaying exponential; 2 = sinewave ');
if type_wave == 1
    t_max = input('Enter duration of exponential: ');
elseif type_wave == 2
    t_max = 1;
end
clf
a = [];
t = 0:.01:t_max;
```

```
cn = tau;
n_max = floor(t_max/tau)
for n = 1:n_max
    LL = (n-1)*tau;
    UL = n*tau;
     if type_wave == 1
        a(n) = (1/cn)*quad(@integrand_exp, LL, UL);
    elseif type_wave == 2
        a(n) = (1/cn)*quad(@integrand_sine, LL, UL);
    end
end
disp(' ')
disp('c_n')
disp(cn)
disp(' ')
disp('Expansion coefficients (approximating pulses not normalized):')
disp(a)
disp(' ')
x_approx = zeros(size(t));
for n = 1:n_max
    x_approx = x_approx + a(n)*phi(t/tau,n-1);
end
if type_wave == 1
    MSE = quad(@integrand_exp2, 0, t_max) - cn*a*a';
elseif type_wave == 2
    MSE = quad(@integrand_sine2, 0, t_max) - cn*a*a';
end
disp('Mean-squared error:')
disp(MSE)
disp(' ')
if type_wave == 1
    plot(t, x_approx), axis([0 t_max -inf inf]), xlabel('t'),
                                ylabel('x(t); x_a_p_p_r_o_x(t)')
    hold
    plot(t, exp(-t))
     title(['Approximation of exp(-t) over [0, ',num2str(t_max),'] with
                contiguous rectangular pulses of width ',num2str(tau)])
elseif type_wave == 2
    plot(t, x_approx), axis([0 t_max -inf inf]), xlabel('t'),
                            ylabel('x(t); x_a_p_p_r_o_x(t)')
```

```
        hold
        plot(t, sin(2*pi*t))
        title(['Approximation of sin(2*pi*t) over [0, ',num2str(t_max),'] with
                        contiguous rectangular pulses of width ',num2str(tau)])
end

% Decaying exponential function for ce_1
%
function z = integrand_exp(t)
z = exp(-t);

% Sine function for ce_1
%
function z = integrand_sine(t)
z = sin(2*pi*t);

% Decaying exponential squared function for ce_1
%
function z = integrand_exp2(t)
z = exp(-2*t);

% Sin^2 function for ce_1
%
function z = integrand_sine2(t)
z = (sin(2*pi*t)).^2;
```

A typical run follows:

```
>> ce2_2
Enter type of waveform: 1 = HR sine; 2 = FR sine; 3 = square; 4 = triangle: 1
```

Computer Exercise 2.3

```
% ce2_4.m: FFT plotting of line spectra for half-rectified, full-rectified sinewave, square wave,
% and triangular waveforms
%
clf
I_wave = input('Enter type of waveform: 1 = positive squarewave; 2 = 0-dc level triangular; 3 = half-rect. sine; 4 = full-wave sine ');
T = 2;
del_t = 0.001;
t = 0:del_t:T;
```

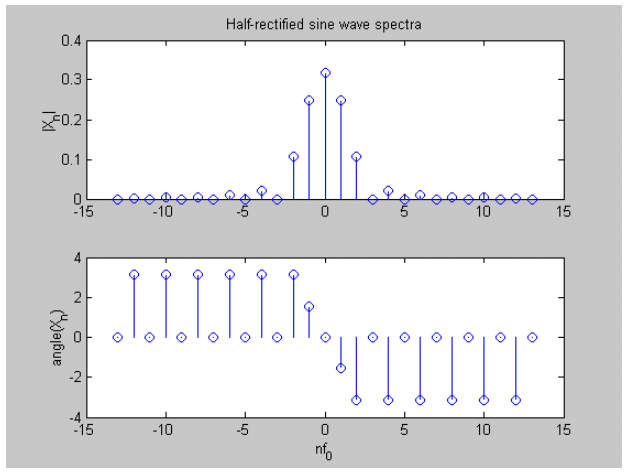

Figure 2.6:

```
L = length(t);
fs = (L-1)/T;
del_f = 1/T;
n = [0 1 2 3 4 5 6 7 8 9];
if I_wave == 1
x = pls_fn(2*(t-T/4)/T);
X_th = abs(0.5*sinc(n/2));
disp(' ')
disp(' 0 - 1 level squarewave')
elseif I_wave == 2
x = 2*trgl_fn(2*(t-T/2)/T)-1;
X_th = 4./(pi^2*n.^2);
X_th(1) = 0; % Set n = even coeficients to zero (odd indexed because of MATLAB)
X_th(3) = 0;
X_th(5) = 0;
X_th(7) = 0;
X_th(9) = 0;
disp(' ')
disp(' 0-dc level triangular wave')
elseif I_wave == 3
x = sin(2*pi*t/T).*pls_fn(2*(t-T/4)/T);
X_th = abs(1./(pi*(1-n.^2)));
```

```
X_th(2) = 0.25;
X_th(4) = 0; % Set n = odd coeficients to zero (even indexed because of MATLAB)
X_th(6) = 0;
X_th(8) = 0;
X_th(10) = 0;
disp(' ')
disp(' Half-rectified sinewave')
elseif I_wave == 4
x = abs(sin(2*pi*t/T));
X_th = abs(2./(pi*(1-n.^2)));
X_th(2) = 0; % Set n = odd coeficients to zero (even indexed because of MATLAB)
X_th(4) = 0;
X_th(6) = 0;
X_th(8) = 0;
X_th(10) = 0;
disp(' ')
disp(' Full-rectified sinewave')
end
X = 0.5*fft(x)*del_t; % Multiply by 0.5 because of 1/T_0 with T_0 = 2
f = 0:del_f:fs;
Y = abs(X(1:10));
Z = [n' Y' X_th'];
disp('Magnetude of the Fourier coefficients');
disp(' ')
disp(' n FFT Theory');
disp(' ___________________________')
disp(' ')
disp(Z);
subplot(2,1,1),plot(t, x), xlabel('t'), ylabel('x(t)')
subplot(2,1,2),plot(f, abs(X),'o'),axis([0 10 0 1]),...
xlabel('n'), ylabel('|X_n|')
```

A typical run follows:

```
>> ce2_3
Enter type of waveform: 1 = positive squarewave; 2 = 0-dc level triangular; 3 = half-rect.
sine; 4 = full-wave sine 3
    Half-rectified sinewave
Magnetude of the Fourier coefficients
n                FFT          Theory
___________________________
```

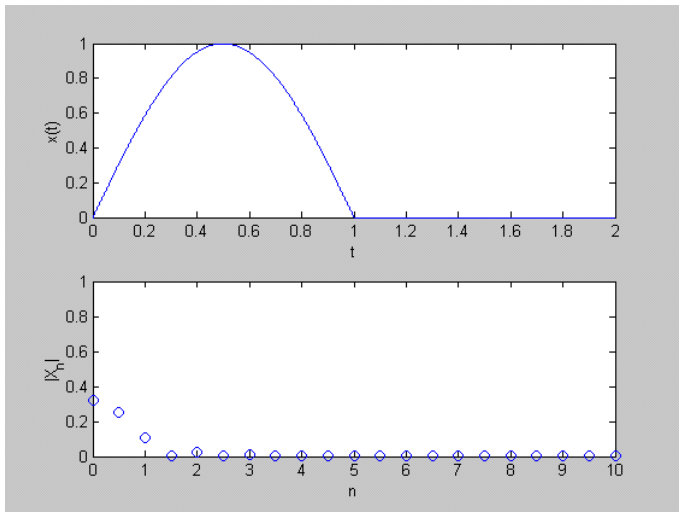

Figure 2.7:

0	0.3183	0.3183
1.0000	0.2501	0.2500
2.0000	0.1062	0.1061
3.0000	0.0001	0
4.0000	0.0212	0.0212
5.0000	0.0001	0
6.0000	0.0091	0.0091
7.0000	0.0000	0
8.0000	0.0051	0.0051
9.0000	0.0000	0

Computer Exercise 2.4

Make the time window long compared with the pulse width.

Computer Exercise 2.5

% ce2_5.m: Finding the energy ratio in a preset bandwidth

%

I_wave = input('Enter type of waveform: 1 = positive squarewave; 2 = triangular; 3 = half-rect. sine; 4 = raised cosine. ');

tau = input('Enter pulse width ');

per_cent = input('Enter percent total desired energy: ');

```
clf
T = 20;
f = [];
G = [];
del_t = 0.01;
t = 0:del_t:T;
L = length(t);
fs = (L-1)/T;
del_f = 1/T;
if I_wave == 1
    x = pls_fn((t-tau/2)/tau);
elseif I_wave == 2
    x = trgl_fn(2*(t-tau/2)/tau);
elseif I_wave == 3
    x = sin(pi*t/tau).*pls_fn((t-tau/2)/tau);
elseif I_wave == 4
    x = abs(sin(pi*t/tau).^2.*pls_fn((t-tau/2)/tau));
end
X = fft(x)*del_t;
f1 = 0:del_f*tau:fs*tau;
G1 = X.*conj(X);
NN = floor(length(G1)/2);
G = G1(1:NN);
ff = f1(1:NN);
f = f1(1:NN+1);
E_tot = sum(G);
E_f = cumsum(G);
E_W = [0 E_f]/E_tot;
test = E_W - per_cent/100;
L_test = length(test);
k = 1;
while test(k) <= 0
    k = k+1;
end
B = k*del_f;
if I_wave == 2
tau1 = tau/2;
else
tau1 = tau;
end
```

```
subplot(3,1,1),plot(t/tau, x), xlabel('t/\tau'), ylabel('x(t)'), axis([0 2 0 1.2])
if I_wave == 1
      title(['Energy containment bandwidth for square pulse of width ',
                                num2str(tau),' seconds'])
elseif I_wave == 2
     title(['Energy containment bandwidth for triangular pulse of width ',
                                num2str(tau),' seconds'])
elseif I_wave == 3
     title(['Energy containment bandwidth for half-sine pulse of width ',
                                num2str(tau),' seconds'])
elseif I_wave == 4
     title(['Energy containment bandwidth for raised cosine pulse of width ',
                                num2str(tau),' seconds'])
end
subplot(3,1,2),semilogy(ff*tau1, abs(G./max(G))), xlabel('f\tau'), ylabel('G'), ...
     axis([0 10 -inf 1])
subplot(3,1,3),plot(f*tau1, E_W,'–'), xlabel('f\tau'), ylabel('E_W'), axis([0 4 0 1.2])
legend([num2str(per_cent), ' bandwidth % X (pulse width) = ', num2str(B*tau)],4)
```

A typical run follows:

```
>> ce2_5
Enter type of waveform: 1 = positive squarewave; 2 = triangular; 3 = half-rect. sine; 4
= raised cosine. 3
Enter pulse width 2
Enter percent total desired energy: 95
```

Computer Exercise 2.6

The program for this exercise is similar to that for Computer Exercise 2.5, except that the waveform is used in the energy calculation.

Computer Exercise 2.7

Use Computer Example 2.2 as a pattern for the solution (note that in earlier printings of the book "Computer Excercise 2.2" should be "Computer Example 2.2").

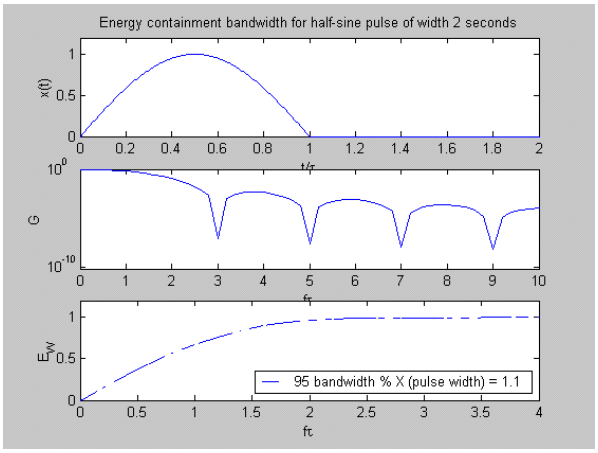
Energy containment bandwidth for half-sine pulse of width 2 seconds
x(t)
t/τ
G
fτ
E_W
95 bandwidth % X (pulse width) = 1.1
fτ

Figure 2.8:

Chapter 3

Basic Modulation Techniques

3.1 Problems

Problem 3.1

The demodulated output, in general, is

$$y_D(t) = \mathrm{Lp}\{x_c(t)\, 2\cos[\omega_c t + \theta(t)]\}$$

where $\mathrm{Lp}\{\bullet\}$ denotes the lowpass portion of the argument. With

$$x_c(t) = A_c m(t) \cos[\omega_c t + \phi_0]$$

the demodulated output becomes

$$y_D(t) = \mathrm{Lp}\{2A_c m(t) \cos[\omega_c t + \phi_0] \cos[\omega_c t + \theta(t)]\}$$

Performing the indicated multiplication and taking the lowpass portion yields

$$y_D(t) = A_c m(t) \cos[\theta(t) - \phi_0]$$

If $\theta(t) = \theta_0$ (a constant), the demodulated output becomes

$$y_D(t) = A_c m(t) \cos[\theta_0 - \phi_0]$$

Letting $A_c = 1$ gives the error

$$\varepsilon(t) = m(t)[1 - \cos(\theta_0 - \phi_0)]$$

The mean-square error is

$$\left\langle \varepsilon^2(t) \right\rangle = \left\langle m^2(t)[1 - \cos(\theta_0 - \phi_0)]^2 \right\rangle$$

where $\langle\cdot\rangle$ denotes the time-average value. Since the term $[1-\cos(\theta_0-\phi_0)]$ is a constant, we have

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)\right\rangle [1-\cos(\theta_0-\phi_0)]^2$$

Note that for $\theta_0=\phi_0$, the demodulation carrier is phase coherent with the original modulation carrier, and the error is zero. For $\theta(t)=\omega_0 t$ we have the demodulated output

$$y_D(t) = A_c m(t)\cos(\omega_0 t-\phi_0)$$

Letting $A_c=1$, for convenience, gives the error

$$\varepsilon(t) = m(t)[1-\cos(\omega_0 t-\phi_0)]$$

giving the mean-square error

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)[1-\cos(\omega_0 t-\phi_0)]^2\right\rangle$$

In many cases, the average of a product is the product of the averages. (We will say more about this in Chapters 4 and 5). For this case

$$\left\langle \varepsilon^2(t)\right\rangle = \left\langle m^2(t)\right\rangle \left\langle [1-\cos(\omega_0 t-\phi_0)]^2\right\rangle$$

Note that $1-\cos(\omega_0 t-\phi_0)$ is periodic. Taking the average over an integer number of periods yields

$$\begin{aligned}\left\langle [1-\cos(\omega_0 t-\phi_0)]^2\right\rangle &= \left\langle 1-2\cos(\omega_0 t-\phi_0)+\cos^2(\omega_0 t-\phi_0)\right\rangle \\ &= 1+\frac{1}{2}=\frac{3}{2}\end{aligned}$$

Thus

$$\left\langle \varepsilon^2(t)\right\rangle = \frac{3}{2}\left\langle m^2(t)\right\rangle$$

Problem 3.2

Multiplying the AM signal

$$x_c(t) = A_c[1+am_n(t)]\cos\omega_c t$$

by $x_c(t) = A_c[1+am_n(t)]\cos\omega_c t$ and lowpass filtering to remove the double frequency $(2\omega_c)$ term yields

$$y_D(t) = A_c[1+am_n(t)]\cos\theta(t)$$

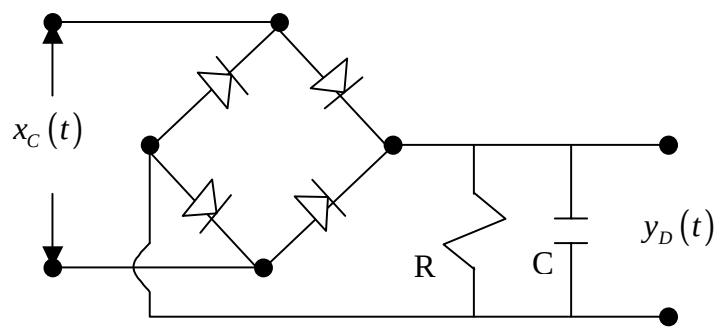

Figure 3.1:

For negligible demodulation phase error, $\theta(t) \approx 0$, this becomes

$$y_D(t) = A_c + A_c a m_n(t)$$

The dc component can be removed resulting in $A_c a m_n(t)$, which is a signal proportional to the message, $m(t)$. This process is not generally used in AM since the reason for using AM is to avoid the necessity for coherent demodulation.

Problem 3.3

A full-wave rectifier takes the form shown in Figure 3.1. The waveforms are shown in Figure 3.2, with the half-wave rectifier on top and the full-wave rectifier on the bottom. The message signal is the envelopes. Decreasing exponentials can be drawn from the peaks of the waveform as depicted in Figure 3.3(b) in the text. It is clear that the full-wave rectified $x_c(t)$ defines the message better than the half-wave rectified $x_c(t)$ since the carrier frequency is effectively doubled.

Problem 3.4

Part	$\langle m_n^2(t) \rangle$	$a = 0.4$	$a = 0.6$	$a = 1$
a	1/3	$E_{ff} = 5,1\%$	$E_{ff} = 10.7\%$	$E_{ff} = 25\%$
b	1/3	$E_{ff} = 5.1\%$	$E_{ff} = 10.7\%$	$E_{ff} = 25\%$
c	1	$E_{ff} = 13.8\%$	$E_{ff} = 26.5\%$	$E_{ff} = 50\%$

Problem 3.5

By inspection, the normalized message signal is as shown in Figure 3.3.
Thus

$$m_n(t) = \frac{2}{T}t, \qquad 0 \le t \le \frac{T}{2}$$

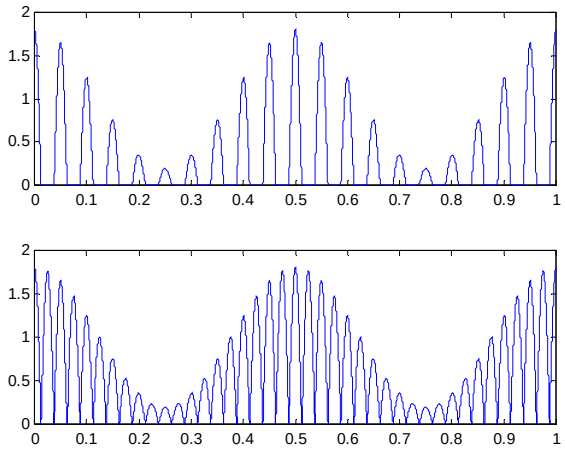

Figure 3.2:

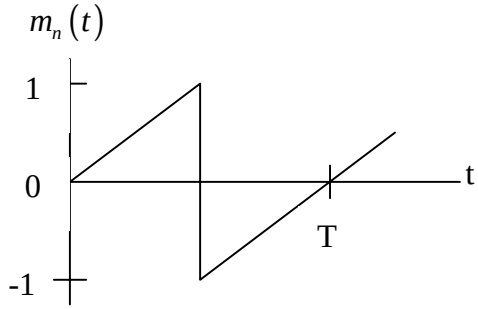

Figure 3.3:

and

$$\left\langle m_n^2(t)\right\rangle = \frac{2}{T}\int_0^{T/2}\left(\frac{2}{T}t\right)^2 dt = \frac{2}{T}\left(\frac{2}{T}\right)^2\frac{1}{3}\left(\frac{T}{2}\right)^3 = \frac{1}{3}$$

Also

$$\begin{aligned} A_c[1+a] &= 40 \\ A_c[1-a] &= 10 \end{aligned}$$

This yields

$$\frac{1+a}{1-a} = \frac{40}{10} = 4$$

or

$$\begin{aligned} 1+a &= 4-4a \\ 5a &= 3 \end{aligned}$$

Thus

$$a = 0.6$$

Since the index is 0.6, we can write

$$A_c[1+0.6] = 40$$

This gives

$$A_c = \frac{40}{1.6} = 25$$

This carrier power is

$$P_c = \frac{1}{2}A_c^2 = \frac{1}{2}(25)^2 = 312.5 \quad \text{Watts}$$

The efficiency is

$$E_{ff} = \frac{(0.6)^2\left(\frac{1}{3}\right)}{1+(0.6)^2\left(\frac{1}{3}\right)} = \frac{0.36}{3.36} = 0.107 = 10.7\%$$

Thus

$$\frac{P_{sb}}{P_c+P_{sb}} = 0.107$$

where P_{sb} represents the power in the sidebands and P_c represents the power in the carrier. The above expression can be written

$$P_{sb} = 0.107 + 0.107P_{sb}$$

This gives

$$P_{sb} = \frac{0.107}{1.0-0.107}P_c = 97.48 \quad \text{Watts}$$

Problem 3.6

For the first signal

$$\tau = \frac{T}{6}, \quad m_n(t) = m(t) \quad \text{and} \quad E_{ff} = 81.25\%$$

and for the second signal

$$\tau = \frac{5T}{6}, \quad m_n(t) = \frac{1}{5}m(t) \quad \text{and} \quad E_{ff} = 14.77\%$$

Problem 3.7

(a) The first step in the solution to this problem is to plot $m(t)$, or use a root-finding algorithm in order to determine the minimum value of $m(t)$. We find that the minimum of $m(t) = -11.9523$, and the minimum falls at $t = 0.0352$ and $t = 0.0648$. Thus the normalized message signal is

$$m_n(t) = \frac{1}{11.9523}\left[9\cos 20\pi t - 7\cos 60\pi t\right]$$

With the given value of $c(t)$ and the index a, we have

$$x_c(t) = 100\left[1 + 0.5m_n(t)\right]\cos 200\pi t$$

This yields

$$\begin{aligned} x_c(t) &= -14.6415\cos 140\pi t + 18.8428\cos 180\pi t \\ &\quad +100\cos 200\pi t \\ &\quad +18.8248\cos 220\pi t - 14.6415\cos 260\pi t \end{aligned}$$

We will need this later to plot the spectrum.
(b) The value of $\langle m_n^2(t)\rangle$ is

$$\langle m_n^2(t)\rangle = \left(\frac{1}{11.9523}\right)^2\left(\frac{1}{2}\right)\left[(9)^2 + (7)^2\right] = 0.455$$

(c)This gives the efficiency

$$E = \frac{(0.5)^2(0.455)}{1 + (0.5)^2(0.455)} = 10.213\%$$

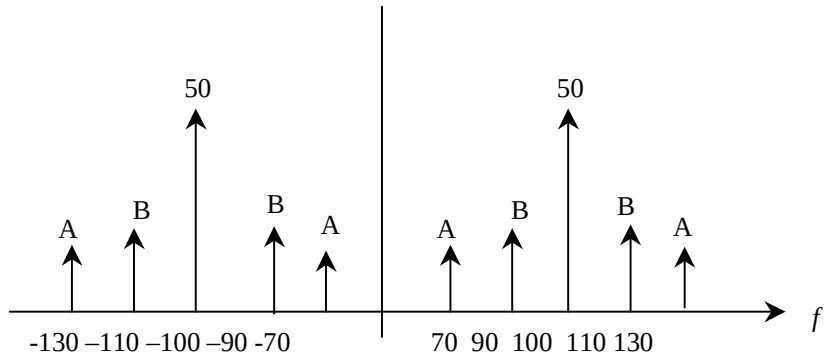

Figure 3.4:

(d) The two-sided amplitude spectrum is shown in Figure 3.4.
where

$$A = \frac{14.4615}{2} = 7.2307$$

and

$$B = \frac{18.8248}{2} = 9.4124$$

The phase spectrum results by noting that A is negative and all other terms are positive.
(e) By inspection the signal is of the form

$$x_c(t) = a(t)c(t)$$

where $a(t)$ is lowpass and $c(t($ is highpass. The spectra of $a(t)$ and $c(t)$ are do not overlap. Thus the Hilbert transform of $c(t)$ is

$$\widehat{x}_c(t) = a(t)\widehat{c}(t)$$

in which $c(t) = 100\cos 200\pi t$. This the envelope is

$$e(t) = 100\sqrt{a^2(t)\cos^2 200\pi t + a^2(t)\sin^2 200\pi t} = 100a(t)$$

where

$$a(t) = [1 + 0.5m_n(t)]$$

Problem 3.8

(a) $m_n(t) = \frac{1}{16}[9\cos 20\pi t + 7\cos 60\pi t]$
(b) $\langle m_n^2(t)\rangle = \left(\frac{1}{16}\right)^2\left(\frac{1}{2}\right)\left[(9)^2 + (7)^2\right] = 0.2539$
(c) $E_{ff} = \frac{0.25(0.2539)}{1+0.25(0.2539)} = 0.05969 = 5.969\%$
(d) The expression for $x_c(t)$ is

$$\begin{aligned} x_c(t) &= 100\left[1 + \frac{1}{32}(9\cos 20\pi t + \cos 60\pi t)\right]\cos 200\pi t \\ &= 10.9375\cos 140\pi t + 14.0625\cos 180\pi t \\ &\quad +100\cos 200\pi t \\ &\quad +14.0625\cos 220\pi t + 10.9375\cos 260\pi t \end{aligned}$$

Note that all terms are positive so the phase spectrum is everywhere zero. The amplitude spectrum is identical to that shown in the previous problem except that

$$\begin{aligned} A &= \frac{1}{2}(10.9375) = 5.46875 \\ B &= \frac{1}{2}(14.0625) = 7.03125 \end{aligned}$$

(e) As in the previous problem, the signal is of the form

$$x_c(t) = a(t)c(t)$$

where $a(t)$ is lowpass and $c(t($ is highpass. The spectra of $a(t)$ and $c(t)$ are do not overlap. Thus the Hilbert transform of $c(t)$ is

$$\widehat{x}_c(t) = a(t)\widehat{c}(t)$$

where $c(t) = 100\cos 200\pi t$. This the envelope is

$$\begin{aligned} e(t) &= 100\sqrt{a^2(t)\cos^2 200\pi t + a^2(t)\sin^2 200\pi t} \\ &= 100a(t) = 100\left[1 + \frac{1}{32}(9\cos 20\pi t + \cos 60\pi t)\right] \end{aligned}$$

Problem 3.9

The modulator output

$$x_c(t) = 40\cos 2\pi(200)t + 4\cos 2\pi(180)t + 4\cos 2\pi(220)t$$

can be written

$$x_c(t) = [40 + 8\cos 2\pi(20)t]\cos 2\pi(200)t$$

or

$$x_c(t) = 40\left[1 + \frac{8}{40}\cos 2\pi(20)t\right]\cos 2\pi(200)t$$

By inspection, the modulation index is

$$a = \frac{8}{40} = 0.2$$

Since the component at 200 Hertz represents the carrier, the carrier power is

$$P_c = \frac{1}{2}(40)^2 = 800 \quad \text{Watts}$$

The components at 180 and 220 Hertz are sideband terms. Thus the sideband power is

$$P_{sb} = \frac{1}{2}(4)^2 + \frac{1}{2}(4)^2 = 16 \quad \text{Watts}$$

Thus, the efficiency is

$$E_{ff} = \frac{P_{sb}}{P_c + P_{sb}} = \frac{16}{800 + 16} = 0.0196 = 1.96\%$$

Problem 3.10

$$A = 14.14 \qquad B = 8.16 \qquad a = 1.1547$$

Problem 3.11

The modulator output

$$x_c(t) = 20\cos 2\pi(150)t + 6\cos 2\pi(160)t + 6\cos 2\pi(140)t$$

is

$$x_c(t) = 20\left[1 + \frac{12}{20}\cos 2\pi(10)t\right]\cos 2\pi(150)t$$

Thus, the modulation index, a, is

$$a = \frac{12}{20} = 0.6$$

The carrier power is

$$P_c = \frac{1}{2}(20)^2 = 200 \quad \text{Watts}$$

and the sideband power is

$$P_{sb} = \frac{1}{2}(6)^2 + \frac{1}{2}(6)^2 = 36 \quad \text{Watts}$$

Thus, the efficiency is

$$E_{ff} = \frac{36}{200+36} = 0.1525$$

Problem 3.12

(a) By plotting $m(t)$ or by using a root-finding algorithm we see that the minimum value of $m(t)$ is $M = -3.432$. Thus

$$m_n(t) = 0.5828\cos(2\pi f_m t) + 0.2914\cos(4\pi f_m t) + 0.5828\cos(10\pi f_m t)$$

The AM signal is

$$\begin{aligned} x_c(t) &= A_c[1 + 0.7m_n(t)]\cos 2\pi f_c t \\ &= 0.2040A_c\cos 2\pi(f_c - 5f_m)t \\ &\quad +0.1020A_c\cos 2\pi(f_c - 2f_m)t \\ &\quad +0.2040A_c\cos 2\pi(f_c - f_m)t \\ &\quad +A_c\cos 2\pi f_c t \\ &\quad +0.2040A_c\cos 2\pi(f_c + f_m)t \\ &\quad +0.1020A_c\cos 2\pi(f_c + 2f_m)t \\ &\quad +0.2040A_c\cos 2\pi(f_c + 5f_m)t \end{aligned}$$

The spectrum is drawn from the expression for $x_c(t)$. It contains 14 discrete components as shown

Comp	Freq	Amp	Comp	Freq	Amp
1	$-f_c - 5f_m$	$0.102A_c$	8	$f_c - 5f_m$	$0.102A_c$
2	$-f_c - 2f_m$	$0.051A_c$	9	$f_c - 2f_m$	$0.051A_c$
3	$-f_c - f_m$	$0.102A_c$	10	$f_c - f_m$	$0.102A_c$
4	$-f_c$	$0.5A_c$	11	f_c	$0.5A_c$
5	$-f_c + f_m$	$0.102A_c$	12	$f_c + f_m$	$0.102A_c$
6	$-f_c + 2f_m$	$0.051A_c$	13	$f_c + 2f_m$	$0.051A_c$
7	$-f_c + 5f_m$	$0.102A_c$	14	$f_c + 5f_m$	$0.102A_c$

(b) The efficiency is 15.8%.

Problem 3.13

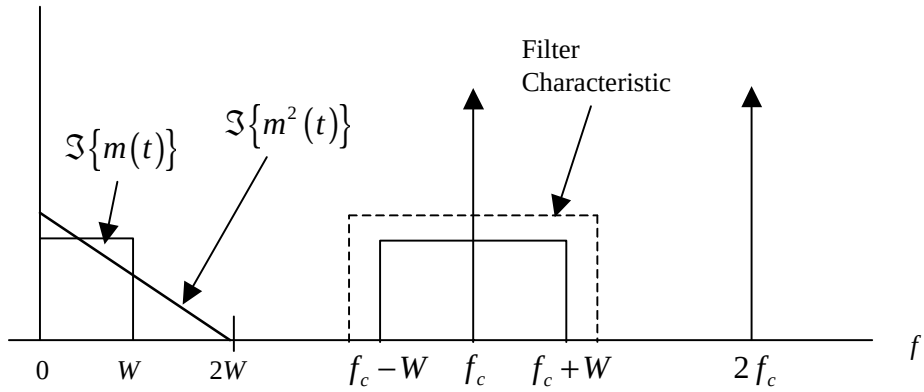

Figure 3.5:

(a) From Figure 3.75

$$x(t) = m(t) + \cos\omega_c t$$

With the given relationship between $x(t)$ and $y(t)$ we can write

$$y(t) = 4\{m(t) + \cos\omega_c t\} + 10\{m(t) + \cos\omega_c t\}^2$$

which can be written

$$y(t) = 4m(t) + 4\cos\omega_c t + 10m^2(t) + 20m(t)\cos\omega_c t + 5 + 5\cos 2\omega_c t$$

The equation for $y(t)$ is more conveniently expressed

$$y(t) = 5 + 4m(t) + 10m^2(t) + 4[1 + 5m(t)]\cos\omega_c t + 5\cos 2\omega_c t$$

(b) The spectrum illustrating the terms involved in $y(t)$ is shown in Figure 3.5. The center frequency of the filter is f_c and the bandwidth must be greater than or equal to $2W$. In addition, $f_c - W > 2W$ or $f_c > 3W$, and $f_c + W < 2f_c$. The last inequality states that $f_c > W$, which is redundant since we know that $f_c > 3W$.
(c) From the definition of $m(t)$ we have

$$m(t) = Mm_n(t)$$

so that

$$g(t) = 4[1 + 5Mm_n(t)]\cos\omega_c t$$

It follows that

$$a = 0.8 = 5M$$

Thus

$$M = \frac{0.8}{5} = 0.16$$

(d) This method of forming a DSB signal avoids the need for a multiplier.

Problem 3.14

$$H_U(f) = 1 - \frac{1}{2}\operatorname{sgn}(f + f_c)$$

$$x_{USB}(t) = \frac{1}{2}A_c m(t)\cos\omega_c t - \frac{1}{2}A_c \widehat{m}(t)\sin\omega_c t$$

Problem 3.15

For the USB SSB case, the modulator output is a sinusoid of frequency $f_c + f_m$, while for the LSB SSB case, the modulator output is a sinusoid of frequency of $f_c - f_m$.

Problem 3.16

Using Figure 3.13 and the phases given in the problem statement, the modulator output becomes

$$\begin{aligned} x_c(t) &= \frac{A\varepsilon}{2}\cos[(\omega_c - \omega_1)t + \phi] \\ &\quad + \frac{A(1-\varepsilon)}{2}\cos[(\omega_c + \omega_1)t + \theta_1] \\ &\quad + \frac{B}{2}\cos[(\omega_c + \omega_2)t + \theta_2] \end{aligned}$$

Multiplying $x_c(t)$ by $4\cos\omega_c t$ and lowpass filtering yields the demodulator output

$$y_D(t) = A\varepsilon\cos(\omega_1 - \phi) + A(1-\varepsilon)\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 t + \theta_2)$$

For the sum of the first two terms to equal the desired output with perhaps a time delay, θ_1 must equal$-\phi$. This gives

$$y_D(t) = A\cos(\omega_1 t + \phi) + B\cos(\omega_2 t + \theta_2)$$

which we can write

$$y_D(t) = A\cos\omega_1\left(t + \frac{\theta_1}{\omega_1}\right) + B\cos\omega_2\left(t + \frac{\theta_2}{\omega_2}\right)$$

For no distortion, $y_D(t)$ must be of the form $m(t-\tau)$. Thus

$$\frac{\theta_1}{\omega_1} = \frac{\theta_2}{\omega_2}$$

so that

$$\theta_2 = \frac{\omega_2}{\omega_1}\theta_1$$

Thus, the phase must be linear. The fact that $\phi = -\theta_1$ tells us that the filter phase response must have odd symmetry about the carrier frequency.

Problem 3.17

We assume that the VSB waveform is given by

$$\begin{aligned} x_c(t) &= \frac{1}{2}A\varepsilon\cos(\omega_c - \omega_1)t \\ &\quad + \frac{1}{2}A(1-\varepsilon)\cos(\omega_c + \omega_1)t \\ &\quad + \frac{1}{2}B\cos(\omega_c + \omega_2)t \end{aligned}$$

We let $y(t)$ be $x_c(t)$ plus a carrier. Thus

$$y(t) = x_c(t) + K\cos\omega_c t$$

It can be shown that $y(t)$ can be written

$$y(t) = y_1(t)\cos\omega_c(t) + y_2(t)\sin\omega_c t$$

where

$$\begin{aligned} y_1(t) &= \frac{A}{2}\cos\omega_1 t + \frac{B}{2}\cos\omega_2 t + K \\ y_2(t) &= \left(A\varepsilon + \frac{A}{2}\right)\sin\omega_1 t - \frac{B}{2}\sin\omega_2 t \end{aligned}$$

Also

$$y(t) = R(t)\cos(\omega_c t + \theta)$$

where $R(t)$ is the envelope and is therefore the output of an envelope detector. It follows that

$$R(t) = \sqrt{y_1^2(t) + y_2^2(t)}$$

For K large, $R(t) = |y_1(t)|$, which is $\frac{1}{2}m(t) + K$, where K is a dc bias. Thus if the detector is ac coupled, K is removed and the output $y(t)$ is $m(t)$ scaled by $\frac{1}{2}$.

Problem 3.18

The required figure appears in Figure 3.6.

Problem 3.19

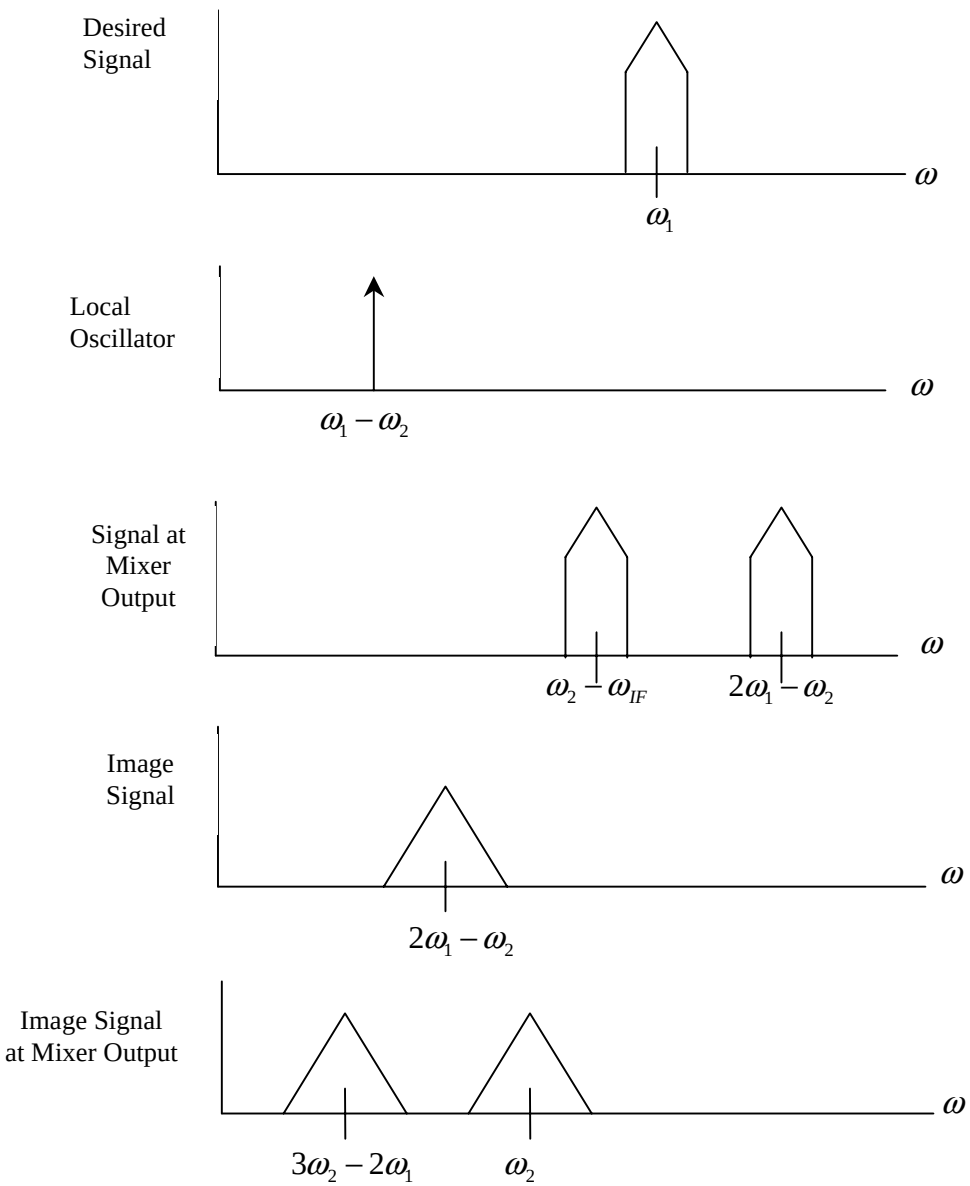
Desired Signal
ω
ω_1
Local Oscillator
ω
$\omega_1 - \omega_2$
Signal at Mixer Output
ω
$\omega_2 - \omega_{IF}$
$2\omega_1 - \omega_2$
Image Signal
ω
$2\omega_1 - \omega_2$
Image Signal at Mixer Output
ω
$3\omega_2 - 2\omega_1$
ω_2

Figure 3.6:

Since high-side tuning is used, the local oscillator frequency is

$$f_{LO} = f_i + f_{IF}$$

where f_i, the carrier frequency of the input signal, varies between 5 and 25 MHz. The ratio is

$$R = \frac{f_{IF} + 25}{f_{IF} + 5}$$

where f_{IF} is the IF frequency expressed in MHz. We make the following table

f_{IF},MHz	R
0.4	4.70
0.5	4.63
0.7	4.51
1.0	4.33
1.5	4.08
2.0	3.86

A plot of R as a function of f_{IF} is the required plot.

Problem 3.20

For high-side tuning we have

$$\begin{aligned} f_{LO} &= f_i + f_{IF} = 1120 + 455 = 1575 \quad \text{kHz} \\ f_{IMAGE} &= f_i + 2f_{IF} = 1120 + 910 = 2030 \quad \text{kHz} \end{aligned}$$

For low-side tuning we have

$$\begin{aligned} f_{LO} &= f_i - f_{IF} = 1120 - 455 = 665 \quad \text{kHz} \\ f_{IMAGE} &= f_i - 2f_{IF} = 1120 - 910 = 210 \quad \text{kHz} \end{aligned}$$

Problem 3.21

For high-side tuning

$$\begin{aligned} f_{LO} &= f_i + f_{IF} = 1120 + 2500 = 3620 \quad \text{kHz} \\ f_{IMAGE} &= f_i + 2f_{IF} = 1120 + 5000 = 6120 \quad \text{kHz} \end{aligned}$$

For low-side tuning

$$\begin{aligned} f_{LO} &= f_i - f_{IF} = 1120 - 2500 = -1380 \quad \text{kHz} \\ f_{LO} &= 1380 \quad kHz \\ f_{IMAGE} &= f_i - 2f_{IF} = 1120 - 5000 = -3880 \quad \text{kHz} \\ f_{LO} &= 3880 \quad kHz \end{aligned}$$

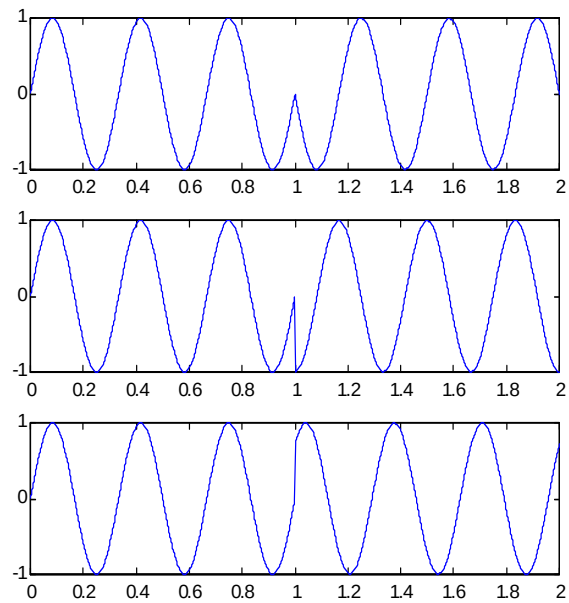

Figure 3.7:

In the preceding development for low-side tuning, recall that the spectra are symmetrical about $f = 0$.

Problem 3.22

By definition

$$x_c(t) = A_c \cos[\omega_c t + k_p m(t)] = A_c \cos[\omega_c t + k_p u(t - t_0)]$$

The waveforms for the three values of k_p are shown in Figure 3.7. The top pane is for $k_p = \pi$, the middle pane is for $k_p = -\pi/2$, and the bottom pane is for $k_p = \pi/4$.

Problem 3.23

Let $\phi(t) = \beta \cos \omega_m t$ where $\omega_m = 2\pi f_m$. This gives

$$x_{c2}(t) = A_c \operatorname{Re}\left\{e^{j\omega_c t} e^{j\beta \cos \omega_m t}\right\}$$

Expanding a Fourier series gives

$$e^{j\beta \cos \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$$

where

$$C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \cos \omega_m t} e^{-jn\omega_m t} dt$$

With $x = \omega_m t$, the Fourier coefficients become

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\beta \cos x} e^{-jnx} dx$$

Since $\cos x = \sin\left(x + \frac{\pi}{2}\right)$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\left[\beta \sin\left(x + \frac{\pi}{2}\right) - nx\right]} dx$$

With $x = x + \frac{\pi}{2}$, the preceding becomes

$$C_n = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} e^{j\left[\beta \sin y - ny + n\frac{\pi}{2}\right]} dy$$

This gives

$$C_n = e^{j\frac{n\pi}{2}} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin y - ny]} dy \right\}$$

where the limits have been adjusted by recognizing that the integrand is periodic with period 2π. Thus

$$C_n = e^{j\frac{n\pi}{2}} J_n(\beta)$$

and

$$x_{c2}(t) = A_c \operatorname{Re}\left\{ e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j\left(n\omega_m t + \frac{n\pi}{2}\right)} \right\}$$

Taking the real part yields

$$x_{c2}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[(\omega_c + n\omega_m) t + \frac{n\pi}{2}\right]$$

The amplitude spectrum is therefore the same as in the preceding problem. The phase spectrum is the same as in the preceding problem except that $\frac{n\pi}{2}$ is added to each term.

Problem 3.24

Since $\sin(x) = \cos(x - \frac{\pi}{2})$ we can write

$$x_{c3}(t) = A_c \sin(\omega_c t + \beta \sin \omega_m t) = A_c \cos\left(\omega_c t - \frac{\pi}{2} + \beta \sin \omega_m t\right)$$

which is

$$x_{c3}(t) = A_c \operatorname{Re}\left\{ e^{j(\omega_c t - \pi/2)} e^{j \sin \omega_m t} \right\}$$

Since

$$e^{j\sin\omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t}$$

we have

$$x_{c3}(t) = A_c \operatorname{Re}\left\{ e^{j(\omega_c t - \pi/2)} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \right\}$$

Thus

$$x_{c3}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \operatorname{Re}\left\{ e^{j(\omega_c t + n\omega_m t - \pi/2)} \right\}$$

Thus

$$x_{c3}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[(\omega_c + n\omega_m) t - \frac{\pi}{2} \right]$$

Note that the amplitude spectrum of $x_{c3}(t)$ is identical to the amplitude spectrum for both $x_{c1}(t)$ and $x_{c2}(t)$. The phase spectrum of $x_{c3}(t)$ is formed from the spectrum of $x_{c1}(t)$ by adding $-\pi/2$ to each term.

For $x_{c4}(t)$ we write

$$x_{c4}(t) = A_c \sin(\omega_c t + \beta \cos\omega_m t) = A_c \cos\left(\omega_c t - \frac{\pi}{2} + \beta \cos\omega_m t \right)$$

Using the result of the preceding problem we write

$$x_{c4}(t) = A_c \operatorname{Re}\left\{ e^{j(\omega_c t - \pi/2)} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j\left(n\omega_m t + \frac{n\pi}{2}\right)} \right\}$$

This gives

$$x_{c4}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \operatorname{Re}\left\{ e^{j(\omega_c t + n\omega_m t - \frac{\pi}{2} + \frac{n\pi}{2})} \right\}$$

Thus

$$x_{c4}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[(\omega_c + n\omega_m) t + \frac{\pi}{2}(n-1) \right]$$

Compared to $x_{c1}(t)$, $x_{c2}(t)$ and $x_{c3}(t)$, we see that the only difference is in the phase spectrum.

Problem 3.25

From the problem statement

$$x_c(t) = A_c \cos\left[2\pi (40) t + 10 \sin(2\pi (5) t) \right]$$

The spectrum is determined using (3.101). Since $f_c = 40$ and $f_m = 5$ (it is important to note that f_c is an integer multiple of f_m) there is considerable overlap of the portion of the spectrum centered about $f = -40$ with the portion of the spectrum centered about $f = 40$ for large β (such as 10). The terms in the spectral plot, normalized with respect to A_c, are given in Table 3.1 for $f \leq 0$ (positive frequency terms and the term at dc). The terms resulting from the portion of the spectrum centered at $f = 40$ are denoted S_{40} and the terms resulting from the portion of the spectrum centered at $f = -40$ are denoted S_{-40}. Given the percision of the Table of Bessel coefficients (Page 131 in the textbook), we have overlap for $|f| \leq 45$. Where terms overlap, they must be summed. The sum is denoted S_T in Table 3.1. In developing the Table 3.1 be sure to remember that $J_{-n}(\beta) = -J_n(\beta)$ for odd n. The power must now be determined in order to find A_c. With the exception of the dc term, the power at each frequency is $S_T^2 A_c^2/2$ (the power for negative frequencies is equal to the power at positive frequencies and so the positive frequency power can simply be doubled to give the total power with the dc term excepted). The power at dc is $S_T^2 A_c^2 = (0.636)^2 A_c^2$. Carring out these operations with the aid of Table 3.1 gives the total power

$$0.8036 A_c^2 = 40$$

which is

$$A_c = \sqrt{\frac{40}{0.8036}} = 7.0552$$

The waveform $x_c(t)$ is illustrated in the top pane of Figure 3.8. The spectrum, normalized with respect to A_c, is illustrated in the bottom frame. The fact that $x_c(t)$ has a nonzero dc value is obvious.

Problem 3.26

We are given $J_0(3) = -0.2601$ and $J_1(3) = -0.3391$. Using

$$J_{n+1}(\beta) = \frac{2n}{\beta} J_n(\beta) - J_{n-1}(\beta)$$

with $\beta = 3$ we have

$$j_{n+1}(3) = \frac{2}{3} n J_n(3) - J_{n-1}(3)$$

With $n = 1$,

$$\begin{aligned} J_2(3) &= \frac{2}{3} J_1(3) - J_0(3) \\ &= \frac{2}{3}(0.3391) + 0.2601 = 0.4862 \end{aligned}$$

Table 3.1: Data for Problem 3.25

f	S_{40}	S_{-40}	S_T
125	0.001	0	0.001
120	0.002	0	0.002
115	0.005	0	0.005
110	0.012	0	0.012
105	0.029	0	0.029
100	0.063	0	0.063
95	0.123	0	0.123
90	0.207	0	0.207
85	0.292	0	0.292
80	0.318	0	0.318
75	0.219	0	0.219
70	−0.014	0	−0.014
65	−0.234	0	−0.234
60	−0.220	0	−0.220
55	0.058	0	0.058
50	0.255	0	0.255
45	0.043	0.001	0.044
40	−0.246	0.002	−0.244
35	−0.043	0.005	−0.038
30	0.255	0.012	0.067
25	−0.058	0.029	−0.029
20	−0.220	0.063	−0.157
15	0.234	0.123	0.357
10	−0.014	0.207	−0.193
5	−0.217	0.292	0.075
0	0.318	0.318	0.636

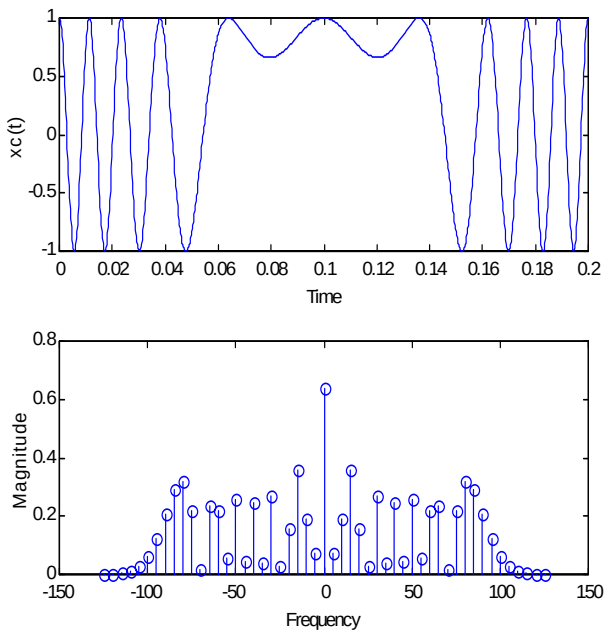

Figure 3.8:

With $n = 2$,

$$\begin{aligned} J_3(3) &= \frac{4}{3} J_2(3) - J_1(3) \\ &= \frac{4}{3}(0.04862) + 0.3391 = 0.3091 \end{aligned}$$

Finally, with $n = 3$ we have

$$\begin{aligned} J_4(3) &= 2J_3(3) - J_2(3) \\ &= 2(0.3091) + 0.4862 = 0.1320 \end{aligned}$$

Problem 3.27

The amplitude and phase spectra follow directly from the table of Fourier-Bessel coefficients. The single-sided magnitude and phase spectra are shown in Figure 3.9. The magnitude spectrum is plotted assuming $A_c = 1$.

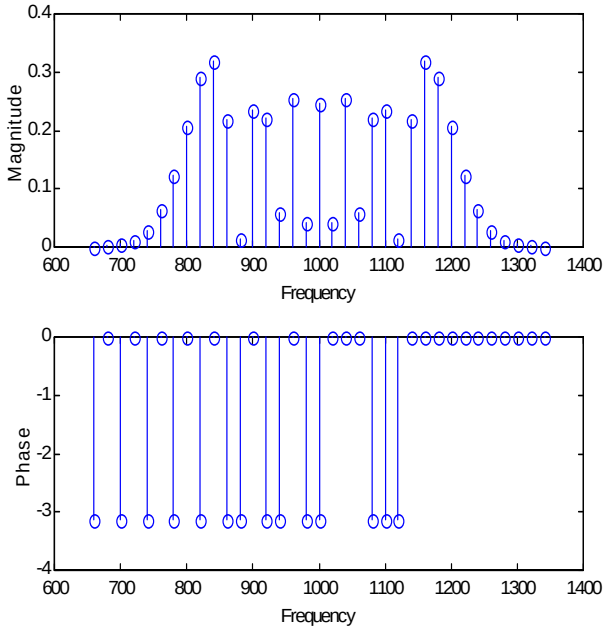
0.4
0.3
0.2
0.1
0
Magnitude
600
700
800
900
1000
1100
1200
1300
1400
Frequency
0
-1
-2
-3
-4
Phase
600
700
800
900
1000
1100
1200
1300
1400
Frequency

Figure 3.9:

Problem 3.28

The modulated signal can be written

$$x_c(t) = \mathrm{Re}\left[10 + 3e^{j2\pi(20)t} + 5e^{-j2\pi(20)t}\right] e^{j2\pi(100)t}$$

We will concentrate on the term in brackets, which is the complex envelope described in Chapter 2. Denoting the complex envelope by $\widetilde{x}_c(t)$, we can write

$$\begin{aligned}\widetilde{x}_c(t) &= [10 + 3\cos 2\pi(20t) + 5\cos 2\pi(20)t] \\ &\quad + j[3\sin 2\pi(20t) - 5\sin 2\pi(20)t] \\ &= [10 + 8\cos 2\pi(20t)] - j[2\sin 2\pi(20)t]\end{aligned}$$

It follows from the definition of $x_c(t)$ that

$$\widetilde{x}_c(t) = R(t)\, e^{j\phi(t)}$$

Thus

$$\begin{aligned}R^2(t) &= [10 + 8\cos 2\pi(20t)]^2 + 4\sin^2 2\pi(20)t \\ &= 134 + 160\cos 2\pi(20)t + 30\cos 2\pi(40)t\end{aligned}$$

This gives

$$R(t) = \sqrt{134 + 160\cos 2\pi(20)t + 30\cos 2\pi(40)t}$$

Also

$$\phi(t) = \tan^{-1}\frac{-2\sin 2\pi(20)t}{10 + 8\cos 2\pi(20)t}$$

Problem 3.29

Since $\sin x = \cos\left(x - \frac{\pi}{2}\right)$, we can write

$$x_c(t) = \mathrm{Re}\left\{e^{j200\pi t}\left[5e^{-j40\pi t} + 10 + 3e^{j\left(40\pi t - \frac{\pi}{2}\right)}\right]\right\}$$

or

$$x_c(t) = \mathrm{Re}\left\{a(t)\, e^{j200\pi t}\right\}$$

where

$$\begin{aligned}a(t) &= (5\cos 40\pi t + 10 + 3\sin 40\pi t) \\ &\quad + j(5\sin 40\pi t - 3\cos 40\pi t)\end{aligned}$$

Thus if we write $x_c(t)$ as

$$x_c(t) = R(t)\cos(200\pi t + \phi(t))$$

the envelope, $R(t)$, is given by

$$R(t) = \left[(5\cos 40\pi t + 10 + 3\sin 40\pi t)^2 + (5\sin 40\pi t - 3\cos 40\pi t)^2\right]^{\frac{1}{2}}$$

and the phase deviation, $\phi(t)$, is given by

$$\phi(t) = \tan^{-1}\frac{5\sin 40\pi t - 3\cos 40\pi t}{10 + 5\cos 40\pi t + 3\sin 40\pi t}$$

The envelope, $R(t)$, can be simplified and expressed in several different forms.

Problem 3.30

(a) Since the carrier frequency is 1000 Hertz, the general form of $x_c(t)$ is

$$x_c(t) = A_c\cos[2\pi(1000)t + \phi(t)]$$

The phase deviation, $\phi(t)$, is therefore given by

$$\phi(t) = 20t^2 \qquad \text{rad}$$

The frequency deviation is

$$\frac{d\phi}{dt} = 40t \qquad \text{rad/sec}$$

or

$$\frac{1}{2\pi}\frac{d\phi}{dt} = \frac{20}{\pi}t \qquad \text{Hertz}$$

(b) The phase deviation is

$$\phi(t) = 2\pi(500)t^2 - 2\pi(1000)t \qquad \text{rad}$$

and the frequency deviation is

$$\begin{aligned}\frac{d\phi}{dt} &= 4\pi(500)t - 2\pi(1000) \qquad \text{rad/sec}\\ &= 2000\pi(t-1) \qquad \text{rad/sec}\end{aligned}$$

or

$$\frac{1}{2\pi}\frac{d\phi}{dt} = 1000(t-1) \qquad \text{Hertz}$$

(c) The phase deviation is

$$\phi(t) = 2\pi(100)t = 200\pi t \qquad \text{rad}$$

and the frequency deviation is

$$\frac{d\phi}{dt} = 200\pi \qquad \text{rad/sec}$$

or

$$\frac{1}{2\pi}\frac{d\phi}{dt} = 100 \qquad \text{Hertz}$$

which should be obvious from the expression for $x_c(t)$.
(d) The phase deviation is

$$\phi(t) = 200\pi t + 10\sqrt{t} \qquad \text{rad}$$

and the phase deviation is

$$\frac{d\phi}{dt} = 200\pi + \frac{1}{2}(10)t^{-\frac{1}{2}} = 200\pi + \frac{5}{\sqrt{t}} \qquad \text{rad/sec}$$

or

$$\frac{1}{2\pi}\frac{d\phi}{dt} = 100 + \frac{5}{2\pi\sqrt{t}} \qquad \text{Hertz}$$

Problem 3.31

(a) The phase deviation is

$$\phi(t) = 2\pi(30)\int_0^t (8)dt = 480\pi t, \qquad t \le 4$$

The maximum phase deviation is $\phi(4) = 480\pi(4) = 1920\pi$. The required plot is simply

$$\phi(t) = \begin{cases} 0, & t < 0 \\ 480\pi t, & 0 \le t < 4 \\ 1920\pi, & t \ge 4 \end{cases}$$

(b) The frequency deviation, in Hz, is

$$\frac{1}{2\pi}\frac{d\phi}{dt} = 30m(t) = \begin{array}{cc} 0, & t < 0 \\ 240, & 0 \le t < 4 \\ 0 & t \ge 4 \end{array}$$

The required sketch follows simply.
(c) The peak frequency deviation is $8f_d = 8\,(30) = 240$ Hertz
(d) The peak phase deviation is

$$2\pi\,(30)\int_0^4 (8)\,dt = 2\pi\,(30)\,(8)\,(4) = 1920\pi \qquad \text{rad}$$

The modulator output power is

$$P = \frac{1}{2}A_c^2 = \frac{1}{2}\,(100)^2 = 5000 \qquad \text{Watts}$$

Problem 3.32

(a) The message signal is

$$m\,(t) = \begin{cases} 0, & t < 4 \\ t-4, & 4 \le t \le 6 \\ 8-t & 6 \le t \le 8 \\ 0, & t > 8 \end{cases}$$

The phase deviation is

$$\begin{aligned} \phi\,(t) &= 2\pi f_d \int_4^t (t-4)\,dt \\ &= 60\pi\,(-4)\,(t-4) + 30\pi\left(t^2-16\right) \\ &= 30\pi\left(t^2-8t+16\right), \quad 4 \le t \le 6 \end{aligned}$$

$$\begin{aligned} \phi\,(t) &= 2\pi f_d \int_6^t (8-t)\,dt + \phi\,(6) \\ &= 120\pi + 60\pi\,(8)\,(t-6) - 30\pi\left(t^2-36\right) \\ &= 120\pi - 30\pi\left(t^2-16t+60\right), \quad 6 \le t \le 8 \end{aligned}$$

Also

$$\begin{aligned} \phi\,(t) &= 240\pi, \quad t > 8 \\ \phi\,(t) &= 0, \quad t < 4 \end{aligned}$$

The sketches follow immediately from the equations.
(b) The frequency deviation in Hertz is $30m(t)$.
(c) The peak phase deviation $= 240\pi$ rad.
(d) The peak frequency deviation $= 120$ Hertz.

(e) The carrier power is, assuming a sufficiently high carrier frequency,

$$P = \frac{1}{2}A_c^2 = \frac{1}{2}(100)^2 = 5000 \qquad \text{Watts}$$

Problem 3.33

The frequency deviation in Hertz is the plot shown in Fig. 3.76 with the ordinate values multiplied by 25.The phase deviation in radians is given

$$\phi(t) = 2\pi f_d \int^t m(\alpha)\, d\alpha = 50\pi \int^t m(\alpha)\, d\alpha$$

For $0 \le t \le 1$, we have

$$\phi(t) = 50\pi \int_0^t 2\alpha d\alpha = 50\pi t^2$$

For $1 \le t \le 2$

$$\begin{aligned} \phi(t) &= \phi(1) + 50\pi \int_1^t (5-\alpha)\, d\alpha = 50\pi + 250\pi(t-1) - 25\pi\left(t^2 - 1\right) \\ &= -175\pi + 250\pi t - 25\pi t^2 \end{aligned}$$

For $2 \le t \le 3$

$$\phi(t) = \phi(2) + 50\pi \int_2^t 3 d\alpha = 225\pi + 150\pi(t-2)$$

For $3 \le t \le 4$

$$\phi(t) = \phi(3) + 50\pi \int_3^t 2 d\alpha = 375\pi + 100\pi(t-3)$$

Finally, for $t > 4$ we recognize that $\phi(t) = \phi(4) = 475\pi$. The required figure results by plotting these curves.

Problem 3.34

The frequency deviation in Hertz is the plot shown in Fig. 3.77 with the ordinate values multiplied by 10. The phase deviation is given by

$$\phi(t) = 2\pi f_d \int^t m(\alpha)\, d\alpha = 20\pi \int^t m(\alpha)\, d\alpha$$

For $0 \le t \le 1$, we have

$$\phi(t) = 20\pi \int_0^t \alpha d\alpha = 10\pi t^2$$

For $1 \leq t \leq 2$

$$\begin{aligned}\phi(t) &= \phi(1) + 20\pi \int_1^t (\alpha - 2)\, d\alpha = 10\pi + 10\pi\left(t^2 - 1\right) - 40\pi(t-1) \\ &= 10\pi\left(t^2 - 4t + 4\right) = 10\pi\,(t-2)^2\end{aligned}$$

For $2 \leq t \leq 4$

$$\begin{aligned}\phi(t) &= \phi(2) + 20\pi \int_2^t (6 - 2\alpha) d\alpha = 0 + 20\pi\,(6)\,(t-2) - 20\pi\left(t^2 - 4\right) \\ &= -20\pi(t^2 - 6t + 8)\end{aligned}$$

Finally, for $t > 4$ we recognize that $\phi(t) = \phi(4) = 0$. The required figure follows by plotting these expressions.

Problem 3.35

The frequency deviation in Hertz is the plot shown in Fig. 3.78 with the ordinate values multiplied by 5. The phase deviation in radians is given by

$$\phi(t) = 2\pi f_d \int^t m(\alpha)\, d\alpha = 10\pi \int^t m(\alpha)\, d\alpha$$

For $0 \leq t \leq 1$, we have

$$\phi(t) = 10\pi \int_0^t (-2\alpha) d\alpha = -10\pi t^2$$

For $1 \leq t \leq 2$

$$\phi(t) = \phi(1) + 10\pi \int_1^t 2 d\alpha = -10\pi + 20\pi\,(t-1) = 10\pi\,(2t-3)$$

For $2 \leq t \leq 2.5$

$$\begin{aligned}\phi(t) &= \phi(2) + 10\pi \int_2^t (10 - 4\alpha) d\alpha = 10\pi + 10\pi\,(10)\,(t-2) - 10\pi(2)\left(t^2 - 4\right) \\ &= 10\pi(-2t^2 + 10t - 11)\end{aligned}$$

For $2.5 \leq t \leq 3$

$$\begin{aligned}\phi(t) &= \phi(2.5) - 10\pi \int_{2.5}^t 2 d\alpha = 15\pi - 20\pi(t - 2.5) \\ &= 10\pi(-2t + 5.5)\end{aligned}$$

For $3 \leq t \leq 4$

$$\begin{aligned}\phi(t) &= \phi(3) + 10\pi \int_3^t (2\alpha - 8)\, d\alpha = 5\pi + 10\pi(t^2 - 9) - 10\pi(8)(t-3) \\ &= 10\pi(t^2 - 8t + 15.5)\end{aligned}$$

Finally, for $t > 4$ we recognize that $\phi(t) = \phi(4) = -5\pi$. The required figure follows by plotting these expressions.

Problem 3.36

(a) The peak deviation is $(12.5)(4) = 50$ and $f_m = 10$. Thus, the modulation index is $\frac{50}{10} = 5$.
(b) The magnitude spectrum is a Fourier-Bessel spectrum with $\beta = 5$. The $n = 0$ term falls at 1000 Hz and the spacing between components is 10 Hz. The sketch is that of Figure 3.24 in the text.
(c) Since β is not $\ll 1$, this is not narrowband FM. The bandwidth exceeds $2f_m$.
(d) For phase modulation, $k_p(4) = 5$ or $k_p = 1.25$.

Problem 3.37

The results are given in the following table:

Part	f_d	$D = 5f_d/W$	$B = 2(D+1)W$
a	20	0.004	50.2 kHz
b	200	0.04	52 kHz
c	2000	0.4	70 kHz
d	20000	4	250 kHz

Problem 3.38

From

$$x_c(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + \omega_m) t$$

we obtain

$$\langle x_c^2(t) \rangle = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

We also know that (assuming that $x_c(t)$ does not have a significant dc component - see Problem 3.24)

$$\langle x_c^2(t) \rangle = \langle A_c^2 \cos^2[\omega_c t + \phi(t)] \rangle$$

which, assuming that $\omega_c \gg 1$ so that $x_c(t)$ has no dc component, is

$$\left\langle x_c^2(t) \right\rangle = \frac{1}{2} A_c^2$$

This gives

$$\frac{1}{2} A_c^2 = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

from which

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

Problem 3.39

Since

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(nx-\beta \sin x)} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

we can write

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\beta \sin x - nx)\, dx + j \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\beta \sin x - nx)\, dx$$

The imaginary part of $J_n(\beta)$ is zero, since the integrand is an odd function of x and the limits $(-\pi, \pi)$ are even. Thus

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\beta \sin x - nx)\, dx$$

Since the integrand is even

$$J_n(\beta) = \frac{1}{\pi} \int_{0}^{\pi} \cos(\beta \sin x - nx)\, dx$$

which is the first required result. With the change of variables $\lambda = \pi - x$, we have

$$\begin{aligned} J_n(\beta) &= \frac{1}{\pi} \int_{0}^{\pi} \cos[\beta \sin(\pi - \lambda) - n(\pi - \lambda)](-1)\, d\lambda \\ &= \frac{1}{\pi} \int_{0}^{\pi} \cos[\beta \sin(\pi - \lambda) - n\pi + n\lambda]\, d\lambda \end{aligned}$$

Since $\sin(\pi - \lambda) = \sin \lambda$, we can write

$$J_n(\beta) = \frac{1}{\pi} \int_{0}^{\pi} \cos[\beta \sin \lambda + n\lambda - n\pi]\, d\lambda$$

Using the identity

$$\cos(u-\nu) = \cos u \cos \nu + \sin u \sin \nu$$

with

$$u = \beta \sin \lambda + n\lambda$$

and

$$\nu = n\pi$$

yields

$$\begin{aligned} J_n(\beta) &= \frac{1}{\pi}\int_0^{\pi} \cos[\beta \sin \lambda + n\lambda] \cos(n\pi)\, d\lambda \\ &\quad + \frac{1}{\pi}\int_0^{\pi} \sin[\beta \sin \lambda + n\lambda] \sin(n\pi)\, d\lambda \end{aligned}$$

Since $\sin(n\pi) = 0$ for all n, the second integral in the preceding expression is zero. Also

$$\cos(n\pi) = (-1)^n$$

Thus

$$J_n(\beta) = (-1)^n \frac{1}{\pi}\int_0^{\pi} \cos[\beta \sin \lambda + n\lambda]\, d\lambda$$

However

$$J_{-n}(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos[\beta \sin \lambda + n\lambda]\, d\lambda$$

Thus

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

or equivalently

$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

Problem 3.40

(a) Peak frequency deviation = 80 Hz
(b) $\phi(t) = 8\sin(20\pi t)$
(c) $\beta = 8$
(d) $P_i = 50$ Watts, $P_0 = 16.76$ Watts
(e) The spectrum of the input signal is a Fourier_Bessel spectrum with $\beta = 8$. The $n = 0$ term is at the carrier frequency of 500 Hz and the spacing between components is 10 Hz. The output spectrum consistes of the $n = 0$ term and three terms each side of the $n = 0$ term. Thus the output spectrum has terms at 470, 480, 490, 500, 510, 520 and 530 Hz.

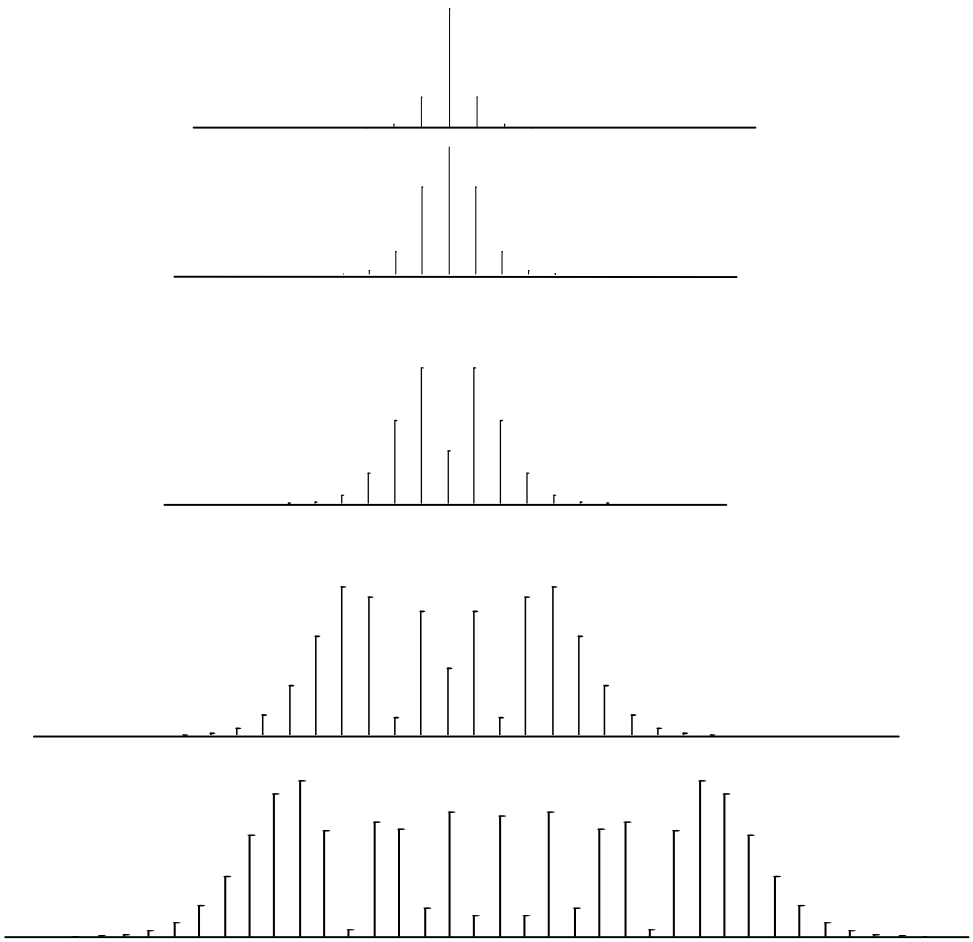

Figure 3.10:

Problem 3.41

The required spectra are given in Figure 3.10. The modulation indices are, from top to bottom, $\beta = 0.5$, $\beta = 1$, $\beta = 2$, $\beta = 5$, and $\beta = 10$.

Problem 3.42

We wish to find k such that

$$P_r = J_0^2(10) + 2\sum_{n=1}^{k} J_0^2(10) \geq 0.80$$

This gives $k = 9$, yielding a power ratio of $P_r = 0.8747$. The bandwidth is therefore

$$B = 2kf_m = 2(9)(150) = 2700 \text{ Hz}$$

For $P_r \geq 0.9$, we have $k = 10$ for a power ratio of 0.9603. This gives

$$B = 2kf_m = 2(10)(150) = 3000 \text{ Hz}$$

Problem 3.43

From the given data, we have

$$f_{c_1} = 110 \quad \text{kHz} \qquad f_{d_1} = 0.05 \qquad f_{d_2} = n(0.05) = 20$$

This gives

$$n = \frac{20}{0.05} = 400$$

and

$$f_{c_1} = n(100) \quad \text{kHz} = 44 \text{ MHz}$$

The two permissible local oscillator frequencies are

$$\begin{aligned} f_{\ell_{0.1}} &= 100 - 44 = 56 \quad \text{MHz} \\ f_{\ell_{0.2}} &= 100 + 44 = 144 \quad \text{MHz} \end{aligned}$$

The center frequency of the bandpass filter must be $f_c = 100$ MHz and the bandwidth is

$$B = 2(D+1)W = 2(20+1)(10)\left(10^3\right)$$

or

$$B = 420 \text{ kHz}$$

Problem 3.44

For the circuit shown

$$H(f) = \frac{E(f)}{X(f)} = \frac{R}{R + j2\pi fL + \frac{1}{j2\pi fC}}$$

or

$$H(f) = \frac{1}{1 + j\left(2\pi f\tau_L - \frac{1}{2\pi f\tau_C}\right)}$$

where

$$\begin{aligned} \tau_L &= \frac{L}{R} = \frac{10^{-3}}{10^3} = 10^{-6}, \\ \tau_C &= RC = \left(10^3\right)\left(10^{-9}\right) = 10^{-6} \end{aligned}$$

A plot of the amplitude response shows that the linear region extends from approximately 54 kHz to118 kHz. Thus an appropriate carrier frequency is

$$f_c = \frac{118 + 54}{2} = 86 \qquad \text{kHz}$$

The slope of the operating characteristic at the operating point is measured from the amplitude response. The result is

$$K_D \cong 8\left(10^{-6}\right)$$

Problem 3.45

We can solve this problem by determining the peak of the amplitude response characteristic. This peak falls at

$$f_p = \frac{1}{2\pi\sqrt{LC}}$$

It is clear that $f_p > 100$ MHz. Let $f_p = 150$ MHz and let $C = 0.001\left(10^{-12}\right)$. This gives

$$L = \frac{1}{(2\pi)^2 f_p^2 C} = 1.126\left(10^{-3}\right)$$

We find the value of R by trial and error using plots of the amplitude response. An appropriate value for R is found to be 1 $M\Omega$. With these values, the discriminator constant is approximately

$$K_D \approx 8.5\left(10^{-9}\right)$$

Problem 3.46

For $A_i = A_c$ we can write, from (3.176),

$$x_r(t) = A_c\left[\cos\omega_c t + \cos(\omega_c + \omega_i)t\right]$$

which is

$$x_r(t) = A_c\left[(1+\cos\omega_i t)\cos\omega_c t - \sin\omega_i t\sin\omega_c t\right]$$

This yields

$$x_r(t) = R(t)\cos\left[\omega_c t + \psi(t)\right]$$

where

$$\psi(t) = \tan^{-1}\left[\frac{\sin\omega_i t}{1+\cos\omega_i t}\right] = \tan^{-1}\left[\tan\frac{\omega_i t}{2}\right] = \frac{\omega_i t}{2}$$

This gives

$$y_D(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\frac{2\pi f_i t}{2}\right) = \frac{1}{2}f_i$$

For $A_i = -A_c$, we get

$$\psi(t) = \tan^{-1}\left[\frac{-\sin\omega_i t}{1-\cos\omega_i t}\right] = -\tan^{-1}\left[\frac{\sin\omega_i t}{1+\cos\omega_i t}\right]$$

Since

$$\frac{\sin x}{1-\cos x} = \cot\frac{x}{2} = \tan\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

we have

$$\psi(t) = \tan^{-1}\left[-\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right] = \frac{x}{2}-\frac{\pi}{2}$$

Thus

$$y_D(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\frac{2\pi f_i t}{2}-\frac{\pi}{2}\right) = \frac{1}{2}f_i$$

Finally, for $A_i \gg A_c$ we see from the phasor diagram that

$$\psi(t) \approx \theta(t) = \omega_i t$$

and

$$y_D(t) = \frac{K_D}{2\pi}\frac{d}{dt}(2\pi f_i t) = f_i$$

Problem 3.47

From Example 3.5, $m(t) = Au(t)$. Thus

$$\phi(t) = k_f\int^t Au(\alpha)\,d\alpha = Ak_f t, \quad t \geq 0$$

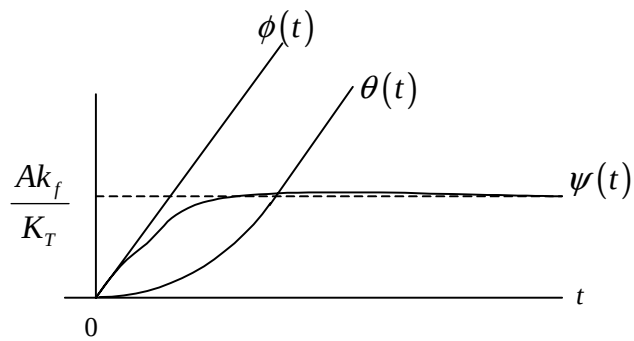

Figure 3.11:

Also

$$\Theta(s) = \frac{AK_T k_f}{s^2(s+K_T)}$$

Taking the inverse transform gives

$$\theta(t) = Ak_f\left(t + \frac{1}{K_T}e^{-K_T t} - \frac{1}{K_T}\right)u(t)$$

The phase error is $\psi(t) = \phi(t) - \theta(t)$. This gives

$$\psi(t) = \frac{Ak_f}{K_T}\left(1 - e^{-K_T t}\right)u(t)$$

The maximum phase error occurs as $t \to \infty$ and is clearly Ak_f/K_T. Thus we require

$$0.2 = \frac{Ak_f}{K_T}$$

and

$$K_T = 5Ak_f$$

The VCO constant is contained in K_T. The required sketch follows.

Problem 3.48

For $m(t) = A\cos\omega_m t$ and

$$\phi(t) = Ak_f\int^t \cos\omega_m\alpha d\alpha = \frac{Ak_f}{\omega_m}\sin\omega_m t$$

and

$$\Phi(s) = \frac{Ak_f}{s^2 + \omega_m^2}$$

The VCO output phase is

$$\Theta(s) = \Phi(s)\frac{K_T}{s + K_T} = \frac{Ak_f K_T}{(s + K_T)(s^2 + \omega_m^2)}$$

Using partial fraction expansion,$\Theta(s)$ can be expressed as

$$\Theta(s) = \frac{Ak_f K_T}{{K_T}^2 + \omega_m^2}\left[\frac{1}{s + K_T} - \frac{s}{s^2 + \omega_m^2} + \frac{K_T}{s^2 + \omega_m^2}\right]$$

This gives, for $t \geq 0$

$$\theta(t) = \frac{Ak_f K_T}{{K_T}^2 + \omega_m^2}\left[e^{-K_T t} - \cos\omega_m t + \frac{K_T}{\omega_m}\sin\omega_m t\right]$$

The first term is the transient response. For large K_T, only the third term is significant. Thus,

$$e_\nu(t) = \frac{1}{K_\nu}\frac{d\theta}{dt} = \frac{Ak_f K_T^2}{K_\nu\left(K_T^2 + \omega_m^2\right)}\cos\omega_m t$$

Also, since K_T is large, $K_T^2 + \omega_m^2 \approx K_T^2$. This gives

$$e_\nu(t) \approx \frac{Ak_f}{K_\nu}\cos\omega_m t$$

and we see that $e_\nu(t)$ is proportional to $m(t)$. If $k_f = K_\nu$, we have

$$e_\nu(t) \approx m(t)$$

Problem 3.49

The Costas PLL is shown in Figure 3.57. The output of the top multiplier is

$$m(t)\cos\omega_c t\left[2\cos(\omega_c t + \theta)\right] = m(t)\cos\theta + m(t)\cos(2\omega_c t + \theta)$$

which, after lowpass filtering, is $m(t)\cos\theta$. The quadrature multiplier output is

$$m(t)\cos\omega_c t\left[2\sin(\omega_c t + \theta)\right] = m(t)\sin\theta + m(t)\sin(2\omega_c t + \theta)$$

which, after lowpass filtering, is $m(t)\sin\theta$. The multiplication of the lowpass filter outputs is

$$m(t)\cos\theta m(t)\sin\theta = m^2(t)\sin 2\theta$$

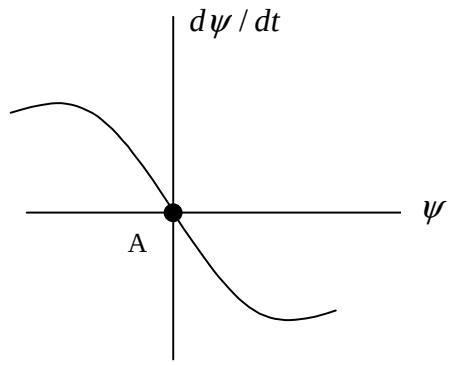

Figure 3.12:

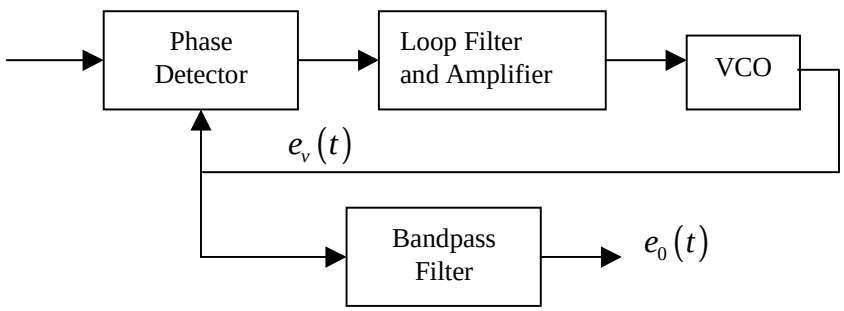

Figure 3.13:

as indicated. Note that with the assumed input $m(t)\cos\omega_c t$ and VCO output $2\cos(\omega_c t+\theta)$, the phase error is θ. Thus the VCO is defined by

$$\frac{d\theta}{dt} = \frac{d\psi}{dt} = -K_\nu e_\nu(t)$$

This is shown below.

Since the $\frac{d\psi}{dt}$ intersection is on a portion of the curve with negative slope, the point A at the origin is a stable operating point. Thus the loop locks with zero phase error and zero frequency error.

Problem 3.50

With $x(t) = A2\pi f_0 t$, we desire $e_0(t) = A\cos 2\pi\left(\frac{7}{3}\right) f_0 t$. Assume that the VCO output is a pulse train with frequency $\frac{1}{3}f_o$. The pulse should be narrow so that the seventh harmonic is relatively large. The spectrum of the VCO output consists of components separated by

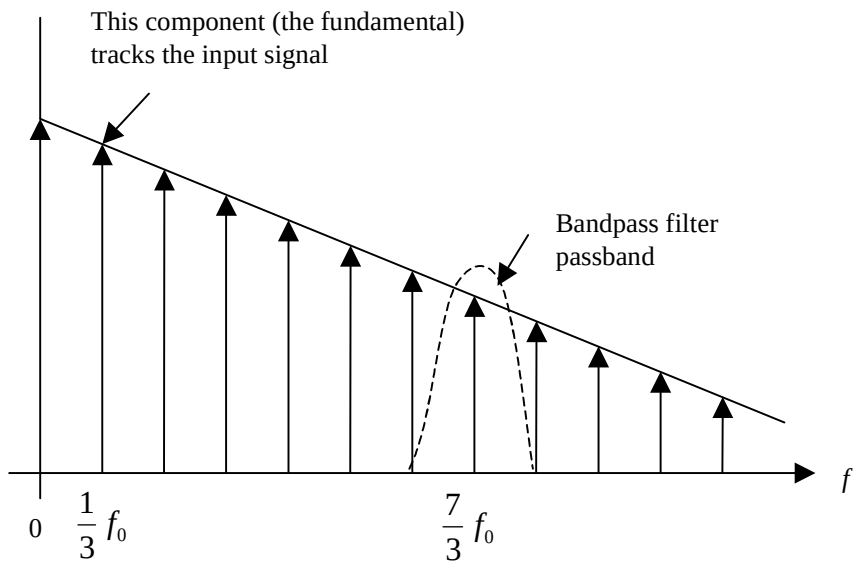

Figure 3.14:

$\frac{1}{3}f_0$ with an envelope of $\text{sinc}(\tau f)$, where τ is the pulse width. The center frequency of the bandpass filter is $\frac{7}{3}f_0$ and the bandwidth is on the order of $\frac{1}{3}f_0$ as shown.

Problem 3.51

The phase plane is defined by

$$\psi = \Delta\omega - K_t \sin\psi(t)$$

at $\psi = 0$, $\psi = \psi_{ss}$, the steady-state phase error. Thus

$$\psi_{ss} = \sin^{-1}\left(\frac{\Delta\omega}{K_t}\right) = \sin^{-1}\left(\frac{\Delta\omega}{2\pi(100)}\right)$$

For $\Delta\omega = 2\pi(30)$

$$\psi_{ss} = \sin^{-1}\left(\frac{30}{100}\right) = 17.46 \text{ degrees}$$

For $\Delta\omega = 2\pi(50)$

$$\psi_{ss} = \sin^{-1}\left(\frac{50}{100}\right) = 30 \text{ degrees}$$

For $\Delta\omega = 2\pi(80)$

$$\psi_{ss} = \sin^{-1}\left(\frac{80}{100}\right) = 53.13 \text{ defrees}$$

For $\Delta\omega = -2\pi\,(80)$

$$\psi_{ss} = \sin^{-1}\left(\frac{-80}{100}\right) = -53.13 \text{ degrees}$$

For $\Delta\omega = 2\pi\,(120)$, there is no stable operating point and the frequency error and the phase error oscillate (PLL slips cycles continually).

Problem 3.52

From (3.228)

$$\frac{\Theta(s)}{\Phi(s)} = \frac{K_t F(s)}{s + K_t F(s)} = \frac{K_t\left(\frac{s+a}{s+\varepsilon}\right)}{s + K_t\left(\frac{s+a}{s+\varepsilon}\right)}$$

which is

$$\frac{\Theta(s)}{\Phi(s)} = \frac{K_t(s+a)}{s(s+\varepsilon) + K_t(s+a)} = \frac{K_t(s+a)}{s^2 + (K_t+\varepsilon)s + K_t a}$$

Therefore

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + (K_t+\varepsilon)s + K_t a$$

This gives

$$\omega_n = \sqrt{K_t a}$$

and

$$\zeta = \frac{K_t+\varepsilon}{2\sqrt{K_t a}}$$

Problem 3.53

Since $\omega_n = 2\pi\,(100)$ we have

$$\sqrt{K_t a} = 2\pi\,(100)$$

or

$$K_t a = 4\pi^2\left(10^4\right)$$

Since

$$\varsigma = \frac{1}{\sqrt{2}} = \frac{K_t+\varepsilon}{2\sqrt{K_t a}} = \frac{1.1K_t}{2\,(2\pi)\,(100)}$$

we have

$$K_t = \frac{\sqrt{2}\,(2\pi)\,(100)}{1.1} = 807.8$$

Thus

$$\varepsilon = 0.1K_t = 80.78$$

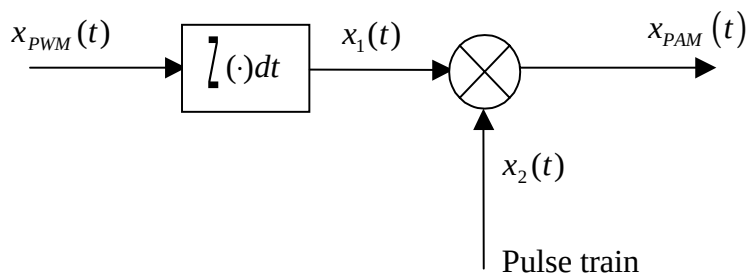

Figure 3.15:

and

$$a = \frac{4\pi^2\left(10^4\right)}{807.8} = 488.7$$

Problem 3.54

From (3.247), we see that the phase deviation is reduced by $1+\frac{1}{2\pi}K_D K_\nu$. The VCO constant is 25 Hz $/volt$. Thus

$$K_\nu = 2\pi\,(25) \qquad \text{rad/s/volt}$$

we may then write

$$\frac{D_1}{D_2} = \frac{5}{0.4} = 12.5 = 1 + \frac{1}{2\pi}K_D\,(2\pi)\,(25)$$

which gives

$$1 + 25K_D = 12.5$$

Thus

$$K_D = 0.46$$

Problem 3.55

A system converting PWM to PAM can be realized as illustrated in Figure 3.15.
The operation should be clear from the waveforms shown in Figure 3.16.
The integrator can be realized by using a capacitor since, for a capacitor,

$$\nu\,(t) = \frac{1}{C}\int i\,(t)\,dt$$

Multiplication by the pulse train can be realized by sampling the capacitor voltage. Thus, the simple circuit is as illustrated in Figure 3.17.

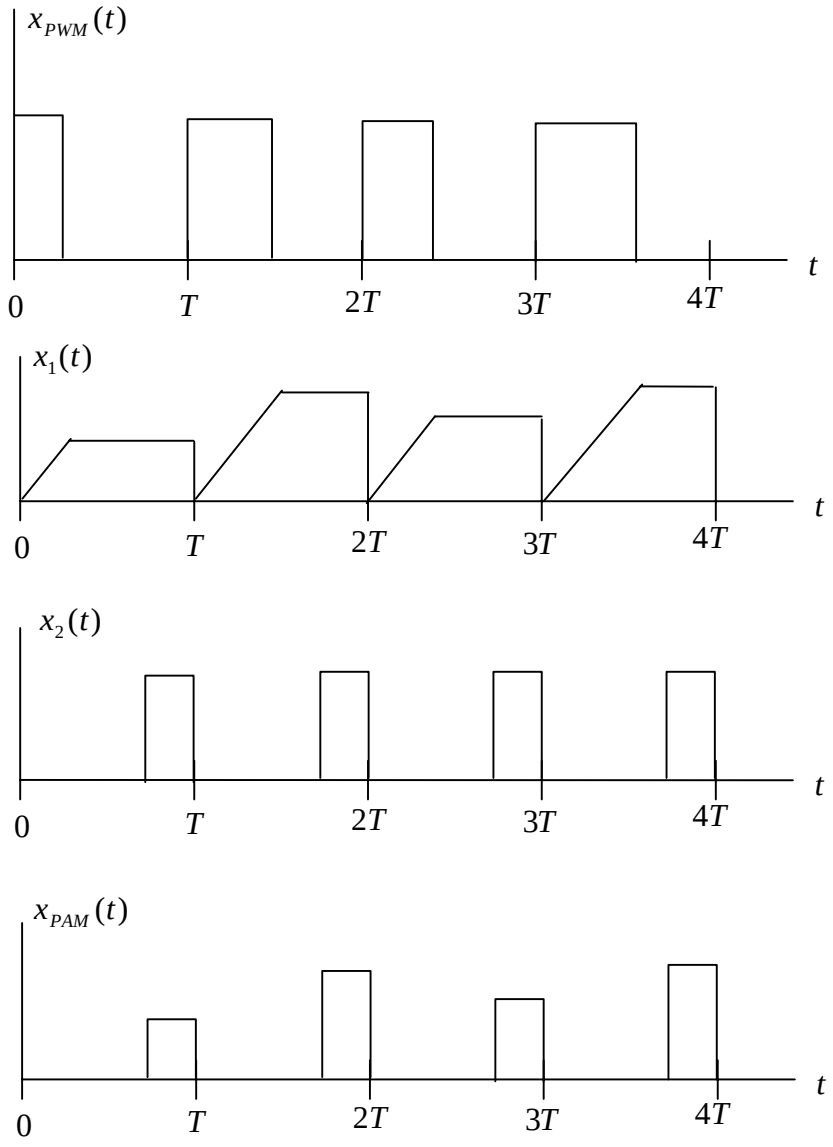
$x_{PWM}(t)$
t
0
T
$2T$
$3T$
$4T$
$x_1(t)$
t
0
T
$2T$
$3T$
$4T$
$x_2(t)$
t
0
T
$2T$
$3T$
$4T$
$x_{PAM}(t)$
t
0
T
$2T$
$3T$
$4T$

Figure 3.16:

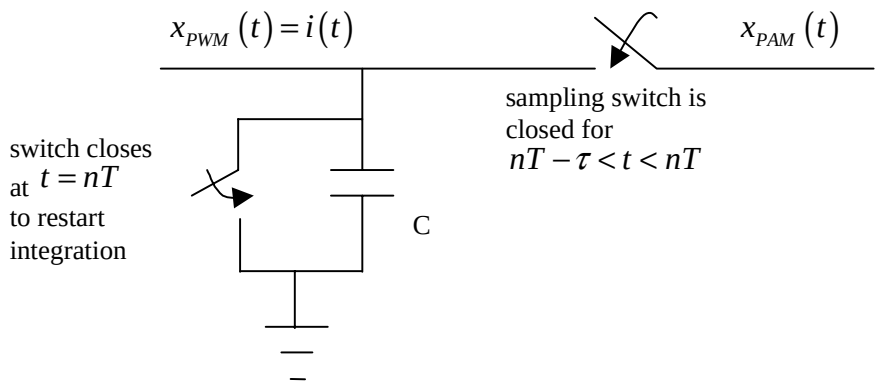

Figure 3.17:

Problem 3.56

Let A be the peak-to-peak value of the data signal. The peak error is 0.5% and the peak-to-peak error is $0.01\,A$. The required number of quantizating levels is

$$\frac{A}{0.01A} = 100 \leq 2^n = q$$

so we choose $q = 128$ and $n = 7$. The bandwidth is

$$B = 2Wk\log_2 q = 2Wk(7)$$

The value of k is estimated by assuming that the speech is sampled at the Nyquist rate. Then the sampling frequency is $f_s = 2W = 8\,\text{kHz}$. Each sample is encoded into $n = 7$ pulses. Let each pulse be τ with corresponding bandwidth $\frac{1}{\tau}$. For our case

$$\tau = \frac{1}{nf_s} = \frac{1}{2Wn}$$

Thus the bandwidth is

$$\frac{1}{\tau} = 2Wn = 2W\log_2 q$$

and so $k = 1$. For $k = 1$

$$B = 2\,(8,000)\,(7) = 112\ \text{kHz}$$

Problem 3.57

Let the maximum error be λA where A is the peak-to-peak message signal. The peak-to-peak error is $2\lambda A$ and the minimum number of quantizing levels

$$q_{\mathsf{min}} = \frac{A}{2\lambda A} = \frac{1}{2\lambda}$$

The wordlength is given by

$$n = \left[\log_2 \frac{1}{2\lambda}\right]$$

where $[x]$ is the smallest integer greater than or equal to x. With $k = 1$ (as in the previous problem), this gives a normalized bandwidth of

$$B_N = \frac{B}{2W} = \left[\log_2 \frac{1}{2\lambda}\right]$$

We make the following table

$\boldsymbol{\lambda}$	$\log\left(\frac{1}{2\lambda}\right)$	B_N
0.001	8.966	9
0.005	6.644	7
0.01	5.644	6
0.05	3.322	4
0.1	2.322	3
0.2	1.322	2
0.4	0.322	1
0.5	0	0

A plot of B_N as a function of λ gives the required plot.

Problem 3.58

The message signal is

$$m(t) = 4\sin 2\pi(10)t + 5\sin 2\pi(20)t$$

The derivative of the message signal is

$$\frac{dm\,(t)}{dt} = 80\pi\cos 2\pi\,(10)\,t + 200\pi\cos 2\pi\,(20t)$$

The maximum value of $dm\,(t)\,/dt$ is obviously 280π and the maximum occurs at $t = 0$. Thus

$$\frac{\delta_0}{T_s} \geq 280\pi$$

or

$$f_s \geq \frac{280\pi}{\delta_0} = \frac{280\pi}{0.05\pi} = 5600$$

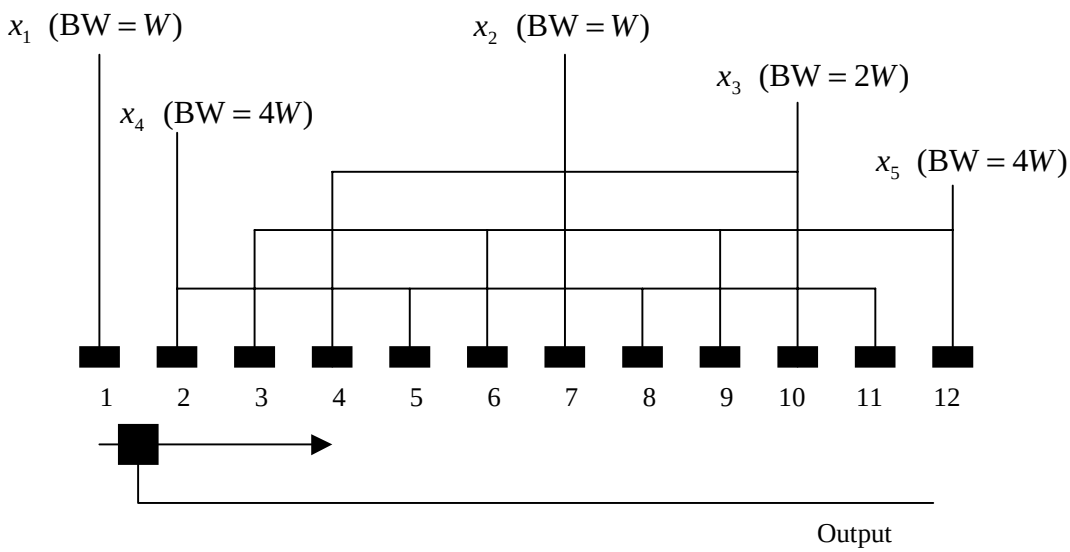

Figure 3.18:

Thus, the minimum sampling frequency is 5600 Hz.

Problem 3.59

One possible commutator configuration is illustrated in Figure 3.18. The signal at the point labeled "output" is the baseband signal. The minimum commutator speed is $2W$ revolutions per second. For simplicity the commutator is laid out in a straight line. Thus, the illustration should be viewed as it would appear wrapped around a cylinder. After taking the sample at point 12 the commutator moves to point 1. On each revolution, the commutator collects 4 samples of x_4 and x_5, 2 samples of x_3, and one sample of x_1 and x_2. The minimum transmission bandwidth is

$$\begin{aligned} B &= \sum_i W_i = W + W + 2W + 4W + 4W \\ &= 12W \end{aligned}$$

Problem 3.60

The single-sided spectrum for $x(t)$ is shown in Figure 3.19.
From the definition of $y(t)$ we have

$$Y(s) = a_1 X(f) + a_2 X(f) * X(f)$$

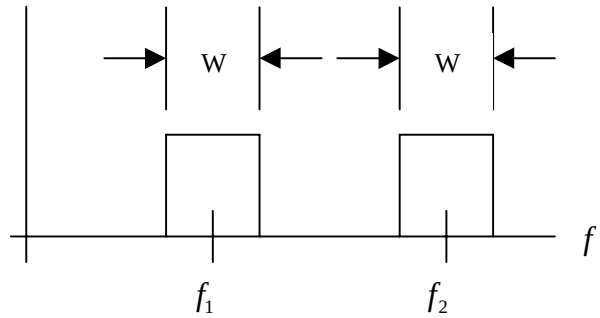

Figure 3.19:

The spectrum for $Y(f)$ is given in Figure 3.20. Demodulation can be a problem since it may be difficult to filter the desired signals from the harmonic and intermodulation distortion caused by the nonlinearity. As more signals are included in $x(t)$, the problem becomes more difficult. The difficulty with harmonically related carriers is that portions of the spectrum of $Y(f)$ are sure to overlap. For example, assume that $f_2 = 2f_1$. For this case, the harmonic distortion arising from the spectrum centered about f_1 falls exactly on top of the spectrum centered about f_2.

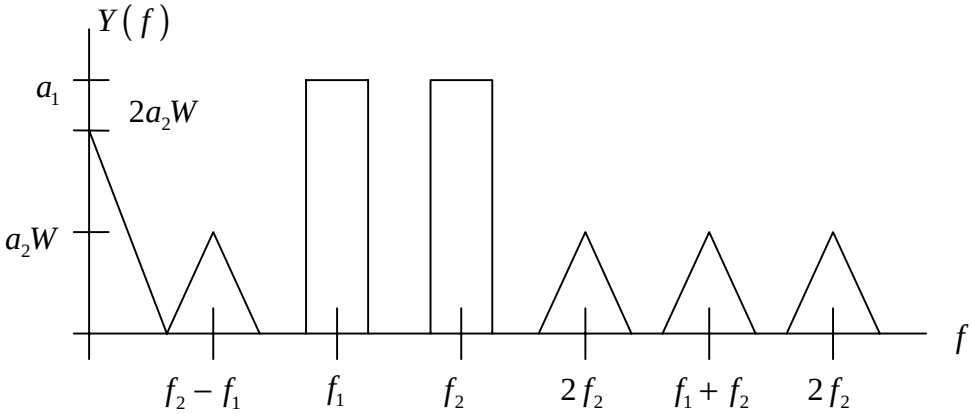

Figure 3.20:

Chapter 4

Probability and Random Variables

4.1 Problem Solutions

Problem 4.1
S = sample space is the collection of the 25 parts. Let A_i = event that the pointer stops on the ith part, $i = 1, 2, ..., 25$. These events are exhaustive and mutually exclusive. Thus

$$P(A_i) = 1/25$$

(a) $P(\text{even number}) = P(2 \text{ or } 4 \text{ or } \cdots \text{ or } 24) = 12/25$;
(b) $P(A_{25}) = 1/25$;
(c) $P(4 \text{ or } 5 \text{ or } 9) = 3/25$;
(d) $P(\text{number} > 10) = P(11, 12, \cdots, \text{ or } 25) = 15/25 = 3/5$.

Problem 4.2
(a) Use a tree diagram similar to Fig. 4.2 to show that

$$\begin{aligned} P(3K, 2A) &= \frac{4}{52}\left\{\frac{4}{51}\left[3\frac{3}{50}\frac{3}{49}\frac{2}{48}\right] + \frac{3}{51}\left[3\frac{4}{50}\frac{3}{49}\frac{2}{48}\right] + \frac{3}{51}\left[\frac{4}{50}\frac{3}{49}\frac{2}{48}\right] + \frac{4}{51}\left[3\frac{3}{50}\frac{3}{49}\frac{2}{48}\right]\right\} \\ &= 9.2345 \times 10^{-6} \end{aligned}$$

(b) Use the tree diagram of Fig. 4.2 except that we now look at outcomes that give 4 of a kind. Since the tree diagram is for a particular denomination, we multiply by 13 and get

$$\begin{aligned} P(4 \text{ of a kind}) &= 13\left\{\frac{4}{52}\left[\frac{3}{51}\left(3\frac{2}{50}\frac{1}{49}\frac{48}{48}\right) + \frac{48}{51}\frac{3}{50}\frac{2}{49}\frac{1}{48}\right] + \frac{48}{52}\frac{4}{51}\frac{3}{50}\frac{2}{49}\frac{1}{48}\right\} \\ &= 0.0002401 \end{aligned}$$

(c) The first card can be anything. Given a particular suit on the first card, the probability of suit match on the second card is 12/51; on the third card it is 11/50; on the fourth card it is 10/49; on the fifth and last card it is 9/48. Therefore,

$$P\left(\text{all same suit}\right) = 1\frac{12}{51}\frac{11}{50}\frac{10}{49}\frac{9}{48} = 0.001981$$

(d) For a royal flush

$$P\left(A, K, Q, J, 10 \text{ of same suit}\right) = \frac{4}{52}\frac{1}{51}\frac{1}{50}\frac{1}{49}\frac{1}{48} = 1.283 \times 10^{-8}$$

(e) The desired probability is

$$P\left(Q|A, K, J, 10 \text{ not all of same suit}\right) = \frac{4}{48} = 0.833$$

Problem 4.3

$$\begin{aligned} P\left(A, B\right) &= P\left(A\right)P\left(B\right) \\ P\left(A, C\right) &= P\left(A\right)P\left(C\right) \\ P\left(B, C\right) &= P\left(B\right)P\left(C\right) \\ P\left(A, B, C\right) &= P\left(A\right)P\left(B\right)P\left(C\right) \end{aligned}$$

Problem 4.4
A and B being mutually exclusive implies that $P\left(A|B\right) = P\left(B|A\right) = 0$. A and B being statistically independent implies that $P\left(A|B\right) = P\left(A\right)$ and $P\left(B|A\right) = P\left(B\right)$. The only way that both of these conditions can be true is for $P\left(A\right) = P\left(B\right) = 0$.

Problem 4.5
(a) The result is

$$P\left(AB\right) = 1 - P\left(1 \text{ or more links broken}\right) = 1 - \left(1 - q^2\right)^2\left(1 - q\right)$$

(b) If link 4 is removed, the result is

$$P\left(AB|\text{ link 4 removed}\right) = 1 - \left(1 - q^2\right)\left(1 - q\right)$$

(c) If link 2 is removed, the result is

$$P\left(AB|\text{ link 2 removed}\right)=1-\left(1-q^{2}\right)^{2}$$

(d) Removal of link 4 is more severe than removal of link 2.

Problem 4.6
Using Bayes' rule

$$P\left(A|B\right)=\frac{P\left(B|A\right)P\left(A\right)}{P\left(B\right)}$$

where, by total probability

$$\begin{aligned}P\left(B\right) &= P\left(B|A\right)P\left(A\right)+P\left(B|\overline{A}\right)P\left(\overline{A}\right)=P\left(B|A\right)P\left(A\right)+\left[1-P\left(\overline{B}|\overline{A}\right)\right]P\left(\overline{A}\right)\\ &= \left(0.9\right)\left(0.4\right)+\left(0.4\right)\left(0.6\right)=0.6\end{aligned}$$

Therefore

$$P\left(A|B\right)=\frac{\left(0.9\right)\left(0.4\right)}{0.6}=0.6$$

Similarly,

$$P\left(A|\overline{B}\right)=\frac{P\left(\overline{B}|A\right)P\left(A\right)}{P\left(\overline{B}\right)}$$

with

$$P\left(\overline{B}\right)=P\left(\overline{B}|A\right)P\left(A\right)+P\left(\overline{B}|\overline{A}\right)P\left(\overline{A}\right)=\left(0.1\right)\left(0.4\right)+\left(0.6\right)\left(0.6\right)=0.4$$

Thus

$$P\left(A|\overline{B}\right)=\frac{\left(0.1\right)\left(0.4\right)}{0.4}=0.1$$

Problem 4.7
(a) $P\left(A_2\right)=0.3$; $P\left(A_2,B_1\right)=0.05$; $P\left(A_1,B_2\right)=0.05$; $P\left(A_3,B_3\right)=0.05$; $P\left(B_1\right)=0.15$; $P\left(B_2\right)=0.25$; $P\left(B_3\right)=0.6$.
(b) $\mathrm{P}(A_3|B_3)=0.083$; $P\left(B_2|A_1\right)=0.091$; $P\left(B_3|A_2\right)=0.333$.

Problem 4.8
See the tables below for the results.

(a)

$\sum$**spots up**	X_1	$P(X_1 = x_i)$
2	4	1/36
3	9	2/36
4	16	3/36
5	25	4/36
6	36	5/36
7	49	6/36
8	64	5/36
9	81	4/36
10	100	3/36
11	121	2/36
12	144	1/36

(b)

$\sum$**spots up**	X_2	$P(X_2 = x_i)$
2	1	
3	1	
4	1	15/36
5	1	
6	1	
7	0	
8	0	
9	0	
10	0	21/36
11	0	
12	0	

Problem 4.9

The cdf is zero for $x < 0$, jumps by 1/8 at $x = 0$, jumps another 3/8 at $x = 1$, jumps another 3/8 at $x = 2$, and jumps another 1/8 at $x = 3$. The pdf consists of an impulse of weight 1/8 at $x = 0$, impulses of weights 3/8 at $x = 1$ and 2, and an impulse of weight 1/8 at $x = 3$.

Problem 4.10

(a) As $x \to \infty$, the cdf approaches 1. Therefore $B = 1$. Assuming continuity of the cdf, $F_x(10) = 1$ also, which says $A \times 10^3 = 1$ or $A = 10^{-3}$.

(b) The pdf is given by

$$f_X(x) = \frac{dF_X(x)}{dx} = 3 \times 10^{-3} x^2 u(x) u(10 - x)$$

The graph of this pdf is 0 for $t < 0$, the quadratic $3 \times 10^{-3} x^2$ for $0 \le x \le 10$, and 0 for $t > 10$.

(c) The desired probability is

$$P(X > 7) = 1 - F_X(7) = 1 - \left(10^{-3}\right)(7)^3 = 1 - 0.343 = 0.657$$

(d) The desired probability is

$$P(3 \le X < 7) = F_X(7) - F_X(3) = 0.316$$

Problem 4.11

(a) $A = \alpha$;
(b) $B = \beta$;
(c) $C = \gamma/2$;
(d) $D = \tau/\pi$

Problem 4.12
(a) Factor the joint pdf as

$$f_{XY}(x,y) = \sqrt{A}\exp[-|x|]\sqrt{A}\exp[-|y|]$$

With proper choice of A, the two separate factors are marginal pdfs. Thus X and Y are statistically independent.
(b) Note that the pdf is the volume between a plane which intersects the x and y coordinate axes one unit out and the f_{XY} coordinate axis at C. First find C from the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1 \text{ or } \int_0^1\int_0^{1-y} C(1-x-y)\,dxdy = 1$$

This gives $C = 6$. Next find $f_X(x)$ and $f_Y(y)$ as

$$f_X(x) = \int_0^{1-x} 6(1-x-y)\,dy = \begin{cases} 3(1-x)^2, 0 \le x \le 1 \\ 0, \text{ otherwise} \end{cases}$$

and

$$f_Y(y) = \int_0^{1-y} 6(1-x-y)\,dx = \begin{cases} 3(1-y)^2, 0 \le y \le 1 \\ 0, \text{ otherwise} \end{cases}$$

Since the joint pdf is not equal to the product of the two marginal pdfs, X and Y are not statistically independent.

Problem 4.13
(a) $\int\int f_{XY}(x,\ y)\,dxdy = 1$ gives $C = 1/255$;
(b) $f_{XY}(0.1, 1.5) = \frac{1+1.5}{255} = 0.0098$
(c) $f_{XY}(x,3) = \begin{cases} (1+3x)/255,\ 0\le x\le 6 \\ 0, \text{ otherwise} \end{cases}$
(d) By integration, $f_Y(y) = \frac{6(1+3y)}{255},\ 0 \le y \le 5$, so the desired conditional pdf is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1+xy}{6(1+3y)},\ 0 \le x \le 6,\ 0 \le y \le 5$$

Substitute $y = 3$ to get the asked for result.

Problem 4.14
(a) To find A, evaluate the double integral

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy &= 1 = \int_0^{\infty}\int_0^{\infty} Axye^{-(x+y)}dxdy \\ &= A\int_0^{\infty} xe^{-x}dx \int_0^{\infty} ye^{-y}dy \\ &= A \end{aligned}$$

Thus $A = 1$.
(b) Clearly, $f_X(x) = xe^{-x}u(x)$ and $f_Y(y) = ye^{-y}u(y)$
(c) Yes, the joint pdf factors into the product of the marginal pdfs.

Problem 4.15
(a) Use normalization of the pdf to unity:

$$\int_{-\infty}^{\infty} \alpha x^{-2} u(x-\alpha)\, dx = \int_{\alpha}^{\infty} \alpha x^{-2} dx = 2x^{-1} \int_{\alpha}^{\infty} = 1$$

Hence, this is the pdf for any value of α.
(b) The cdf is

$$F_X(x) = \int_{-\infty}^{x} \alpha z^{-2} u(z-\alpha)\, dz = \begin{cases} 0, \ x < \alpha \\ (1-\alpha/x), \ x > \alpha \end{cases}$$

(c) The desired probability is

$$P(X \geq 10) = 1 - P(X < 10) = 1 - F_X(10) = \begin{cases} 1, \ \alpha > 10 \\ \alpha/10, \ \alpha < 10 \end{cases}$$

Problem 4.16
The result is

$$f_Y(y) = \begin{cases} \frac{\exp\left(-y/2\sigma^2\right)}{\sqrt{2\pi\sigma^2 y}}, \ y \geq 0 \\ 0, \ y < 0 \end{cases}$$

Problem 4.17
First note that

$$P(Y = 0) = P(X \leq 0) = 1/2$$

For $y > 0$, transformation of variables gives

$$f_Y(y) = f_X(x) \left| \frac{dg^{-1}(y)}{dy} \right|_{x=g^{-1}(y)}$$

Since $y = g(x) = ax$, we have $g^{-1}(y) = y/a$. Therefore

$$f_Y(y) = \frac{\exp\left(-\frac{y^2}{2a^2\sigma^2}\right)}{\sqrt{2\pi a^2 \sigma^2}}, \ y > 0$$

For $y = 0$, we need to add $0.5\delta(y)$ to reflect the fact that Y takes on the value 0 with probability 0.5. Hence, for all y, the result is

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{\exp\left(-\frac{y^2}{2a^2\sigma^2}\right)}{\sqrt{2\pi a^2\sigma^2}}u(y)$$

where $u(y)$ is the unit step function.

Problem 4.18
(a) The normalization of the pdf to unity provides the relationship

$$A\int_{-\infty}^{\infty} e^{-b|x|}dx = 2A\int_0^{\infty} e^{-bx}dx = 2A/b = 1$$

where the second integral follows because of evenness of the integrand. Thus $A = b/2$.
(b) $E[X] = 0$ because the pdf is an even function of x.
(c) Since the expectation of X is zero,

$$\sigma_X^2 = E\left\{X^2\right\} = \int_{-\infty}^{\infty} \frac{b}{2}x^2 e^{-b|x|}dx = b\int_0^{\infty} x^2 e^{-bx}dx = 2/b^2$$

where evenness of the integrand has again been used, and the last integral can be found in an integral table.

Problem 4.19
Use the fact that $E\left\{[X - E(X)]^2\right\} \geq 0$ (0 only if $X = 0$ with probability one). Expanding, we have $E\left[X^2\right] - \{E[X]\}^2 > 0$ if $X \neq 0$.

Problem 4.20
For the uniform distribution, see Example 4.21. The mean for the Gaussian pdf follows by a change of variables and the normalization property of the Gaussian pdf. Its variance follows by defining the random variable $Y = X - m$, which has zero mean, and using a tabulated definite integral. The mean and variance of the Rayleigh random variable can be obtained by using tabulated definite integrals. The mean of a Laplacian random variable is zero by the evenness of the pdf. The variance was obtained in Problem 4.18. The mean and variance of the single-sided exponential pdf are easily found in terms of tabulated integrals. The mean of the hyperbolic pdf is zero by virtue of the evenness of the pdf, and its variance is worked out below. The mean and variance of the Nakagami-m pdf can be put in terms of tabulated definite integrals. The mean and variance of the binomial random variable is worked out beginning with (4.162). The mean and variance for the Poisson and geometric distributions follow similarly. For the variance of the hyperbolic pdf, consider

$$E\left[X^2\right] = \int_{-\infty}^{\infty} \frac{x^2(m-1)h^{m-1}dx}{2(|x|+h)^m}dx = \int_0^{\infty} \frac{x^2(m-1)h^{m-1}dx}{2(x+h)^m}dx$$

where the second integral follows by evenness of the integrand in the first integral, and the absolute value on x is unnecessary because the integration is for positive x. Integrate by parts twice to obtain

$$E\left[X^2\right] = (m-1)\,h^{m-1}\left\{\left[\frac{x^2\,(h+h)^{-m-1}}{-m+1}\right]_0^{\infty} + \int_0^{\infty}\frac{2x\,(x+h)^{-m+1}}{m-1}dx\right\}$$

The first term is zero if $m \geq 3$. Therefore

$$E\left[X^2\right] = 2h^{m-1}\int_0^{\infty} x\,(x+h)^{-m+1}\,dx = \frac{2h^2}{(m-2)\,(m-3)},\ m \geq 4$$

Problem 4.21
(a) $A = b\left[1-e^{-bB}\right]^{-1}$;
(b) The cdf is 0 for $x < 0$ and 1 for $x > B$. For $0 \leq x \leq B$, the result is $(A/b)\left(1-e^{-bx}\right)$;
(c) The mean is

$$E\left[X\right] = \frac{1}{b}\left[1 - bB\frac{e^{-bB}}{1-e^{-bB}}\right]$$

(d) The mean-square is

$$E\left[X^2\right] = \frac{2A}{b}\left[\frac{1}{b^2} - \frac{e^{-bB}}{b^2}\,(1+bB)\right] - \frac{AB^2}{b}e^{-bB}$$

where A must be substituted from part (a.).
(e) For the variance, subtract the result of part (c) squared from the result of part (d).

Problem 4.22
(a) The integral evaluates as follows:

$$E\left\{X^{2n}\right\} = \int_{-\infty}^{\infty} x^{2n}\frac{e^{-x2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}dx = 2\int_0^{\infty}\left(2\sqrt{2}\sigma y\right)^{2n}\frac{e^{-y^2}}{\sqrt{\pi}}dy = \frac{2^{n+1}\sigma^{2n}}{\sqrt{\pi}}\int_0^{\infty} y^{2n}e^{-y}dy$$

where $y = x/\left(2^{1/2}\sigma\right)$. Using a table of definite integrals, the given result is obtained.
(b) The integral is zero by virtue of the oddness of the integrand.

Problem 4.23

$$\begin{aligned} E\left[X\right] &= \int_{-\infty}^{\infty} x\left\{\frac{1}{2}f\,(x-4) + \frac{1}{8}\left[u\,(x-3) - u\,(x-7)\right]\right\}dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} xf\,(x-4)\,dx + \frac{1}{8}\int_3^7 xdx = \frac{9}{2} \end{aligned}$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \left\{\frac{1}{2} f(x-4) + \frac{1}{8}\left[u(x-3) - u(x-7)\right]\right\} dx = \frac{127}{6}$$

$$\alpha_X^2 = E\left[X^2\right] - E^2[X] = \frac{127}{6} - \frac{81}{4} = \frac{11}{12}$$

Problem 4.24
Regardless of the correlation coefficient, the mean of Z is

$$E\{Z\} = E\{3X - 4Y\} = 3E\{X\} - 4E\{Y\} = 3(2) - 4(1) = 2$$

The variance of Z is

$$\begin{aligned}
\operatorname{var}(Z) &= E\left\{[Z - E(Z)]^2\right\} \\
&= E\left\{[3X - 4Y - 3E(X) + 4E(Y)]^2\right\} \\
&= E\left\{\left[3\left(X - \overline{X}\right) - 4(Y - \overline{Y})\right]^2\right\} \\
&= E\left\{9\left(X - \overline{X}\right) - 24\left(X - \overline{X}\right)(Y - \overline{Y}) + 16(Y - \overline{Y})^2\right\} \\
&= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_X\sigma_Y\sigma_{XY}
\end{aligned}$$

Putting in numbers, the results for the variance of Z are: (a) 107; (b) 83.76; (c) 51.23; (d) 14.05.

Problem 4.25
Consider independent Gaussian random variables U and V and define a transformation

$$g_1(u,v) = x = \rho u + \sqrt{1-\rho^2}u \text{ and } g_2(u,v) = y = u$$

The inverse transformation is $u = y$ and $v = (x - \rho y)/\left(1-\rho^2\right)^{1/2}$. Since the transformation is linear, the new random variables are Gaussian. The Jacobian is $\left(1-\rho^2\right)^{-1/2}$. Thus the joint pdf of the new random variables X and Y is

$$f_{XY}(x,y) = \frac{1}{\sqrt{1-\rho^2}} \frac{e^{-\frac{u^2+v^2}{2\sigma^2}}}{2\pi\sigma^2}\Bigg|_{\substack{u=g_1^{-1}(x,y)\\ v=g_2^{-1}(x,y)}} = \frac{\exp\left[-\frac{x^2-2\rho xy+y^2}{2\sigma^2(1-\rho^2)}\right]}{2\pi\sigma^2(1-\rho^2)}$$

Thus X and Y are equivalent to the random variables in the problem statement and which proves the desired result. To see this, note that

$$E\{XY\} = E\left\{\left(\rho U + \sqrt{1-\rho^2}V\right)U\right\} = \rho\sigma^2$$

which proves the desired result.

Problem 4.26
Divide the joint Gaussian pdf of two random variables by the Gaussisn pdf of Y, collect all exponential terms in a common exponent, complete the square of this exponent which will be quadratic in x and y, and the desired result is a Gaussian pdf with

$$E\{X|Y\} = m_x + \frac{\rho\sigma_x}{\sigma_Y}(Y - m_Y) \text{ and } \operatorname{var}(X|Y) = \sigma_X^2\left(1-\rho^2\right)$$

Problem 4.27

Convolve the two component pdfs. A sketch of the two component pdfs making up the integrand show that there is no overlap until $z > -1/2$, and the overlap ends when $z > 3.5$. The result, either obtained graphically or analytically, is given by

$$f_Z(z) = \begin{cases} 0, \ z < -0.5 \\ (z+0.5)/3, \ -0.5 \le z \le 0.5 \\ 1/3, \ 0.5 \le z < 2.5 \\ -(z-3.5)/3, \ 2.5 \le z \le 3.5 \\ 0, \ z > 3.5 \end{cases}$$

Problem 4.28

(a) $E[X] = 0, E\left[X^2\right] = 1/32 = \text{var}[X]$;
(b) The pdf of Y is

$$f_Y(y) = \frac{4}{3}e^{-8|y-3|/3}$$

(c)

$$\begin{aligned} E[Y] &= E[2+3X] = 2, \\ E[Y] &= E\left[(2+3X)^2\right] \\ &= E\left[4+12X+9X^2\right] = 4+12E[X]+9E\left[X^2\right] \\ &= 4+12+9/32 = 4.28125, \\ \sigma^2 &= 4.28125 - 2^2 = 0.28125 \end{aligned}$$

Problem 4.29
(a) The characteristic function is

$$M_X(jv) = \frac{a}{a-jv}$$

(b) The mean and mean-square values are $E[X] = 1/a$ and $E\left[X^2\right] = 2/a^2$.
(c) $\text{var}[X] = 1/a^2$.

Problem 4.30
The required distributions are

$$\begin{aligned} P(k) &= \binom{n}{k} p^k (1-p)^{n-k} \text{ (Binomial)} \\ P(k) &= \frac{e^{-(k-np)^2/[2np(1-p)]}}{\sqrt{2\pi np(1-p)}} \text{ (Laplace)} \\ P(k) &= \frac{(np)^k}{k!} e^{-np} \text{ (Poisson)} \end{aligned}$$

A comparison is given below:

(a)

n, p	k	**Binomial**	**Laplace**	**Poisson**
3,1/5	0	0.512	0.396	0.549
	1	0.384	0.487	0.329
	2	0.096	0.075	0.099
	3	0.008	0.0014	0.020

(b)

n, p	k	**Binomial**	**Laplace**	**Poisson**
3,1/10	0	0.729	0.650	0.741
	1	0.243	0.310	0.222
	2	0.027	0.004	0.033
	3	0.001	1×10^{-6}	0.003

(c)

n, p	k	**Binomial**	**Laplace**	**Poisson**
10,1/5	0	0.107	0.090	0.135
	1	0.268	0.231	0.271
	2	0.302	0.315	0.271
	3	0.201	0.231	0.180
	4	0.088	0.090	0.090
	5	0.026	0.019	0.036
	6	0.005	0.002	0.012
	7	7.9×10^{-4}	1.3×10^{-4}	3.4×10^{-3}
	8	7.0×10^{-5}	4.1×10^{-6}	8.6×10^{-4}
	9	4.1×10^{-6}	7.1×10^{-8}	1.9×10^{-4}

(d)

n, p	k	**Binomial**	**Laplace**	**Poisson**
10,1/10	0	0.348	0.241	0.368
	1	0.387	0.420	0.368
	2	0.194	0.241	0.183
	3	0.057	0.046	0.061
	4	0.011	0.003	0.015
	5	1.5×10^{-3}	6×10^{-5}	3.1×10^{-3}
	6	1.4×10^{-4}	3.9×10^{-7}	5.1×10^{-4}
	7	3.6×10^{-7}	6.3×10^{-13}	3.4×10^{-3}
	8	9.0×10^{-9}	1.5×10^{-16}	1×10^{-6}
	9	1×10^{-10}	1.2×10^{-20}	1×10^{-7}

Problem 4.31
(a) P (5 or 6 heads in 20 trials) $= \sum_{k=5,6} [20!/k!\,(20-k)!]\,2^{-20} = 0.0518$
(b) P (first head at trial 15) $= (1/2)(1/2)^{14} = 3.052 \times 10^{-5}$
(c) P (50 to 60 heads in 100 trials) $= \sum_{k=50}^{60} \frac{100!}{k!(100-k)!} 2^{-100}$

Problem 4.32
(a) $N_p = 26\,(25)\,(24)\,(21) = 358,800$;
(b) $N_p = (26)^4 = 456,976$;
(c) The answers are the reciprocals of the numbers found in (a) and (b).

Problem 4.33
(a) The desired probability is

$$\begin{aligned} P \text{ (fewer than 3 heads in 10 tosses)} &= \sum_{k=0}^{2} \binom{10}{k} \left(\frac{1}{2}\right)^{10} \\ &= (1+10+45)\,2^{-10} = 0.0547 \end{aligned}$$

(b) $npq = 10\,(1/2)\,(1/2) = 2.25$; $np = 5$. The Laplace-approximated probability is

$$\begin{aligned} P(< 3 \text{ heads in 10 tosses}) &= \sum_{k=0}^{2} \frac{e^{-(k-5)^2/5}}{\sqrt{5\pi}} \\ &= \frac{1}{\sqrt{5\pi}} \left(e^{-5} + e^{-16/5} + e^{-9/5}\right) = 0.0537 \end{aligned}$$

(c) The percent error is

$$\%\text{ error } = \frac{0,0547 - 0.0537}{0.0547} \times 10^2 = 1.84\%$$

Problem 4.34

P (3 or more errors in 10^6) $= 0.08$

Problem 4.35

First note that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} du = \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(u-b)^2} du = \sqrt{\frac{2\pi}{a}}$$

(a) In the joint Gaussian pdf, let

$$u = \frac{x - m_X}{\sigma_X};\ v = \frac{y - m_Y}{\sigma_Y};\ k = \left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right)^{-1}$$

Then the joint Gaussian pdf becomes

$$f_{XY}(x,y) = k\exp\left[-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right]$$

Complete the square in v to obtain

$$f_{XY}(x,y) = k\exp\left[-\frac{(v-2\rho u)^2}{2(1-\rho^2)}\right]\exp\left(-u^2/2\right)$$

Use the definite integral above to integrate over v with the result

$$f_X(x) = \sqrt{2\pi(1-\rho^2)\sigma_Y^2}\exp\left(-u^2/2\right) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left[-\frac{(x-m_X)^2}{2\sigma_X^2}\right]$$

The pdf of Y can be found in a similar manner.
(b) First find the characteristic function:

$$M_X(jv) = E\left\{e^{jvX}\right\} = \int_{-\infty}^{\infty} e^{jvx}\frac{e^{-\frac{(x-m_X)^2}{2\sigma_X}}}{\sqrt{2\pi\sigma_X^2}}dx$$

Put the exponents together and complete the square in the exponent to get

$$M_X(jv) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left[-\frac{1}{2}\sigma_X^2 v^2 + jm_X v\right]\exp\left[-\frac{1}{2\sigma_X^2}\left(x - m_X - jv\sigma_X^2\right)^2\right]dx$$

Use the definite integral given above to get

$$M_X(jv) = \exp\left[jvm_X - \frac{1}{2}\sigma_X^2 v^2\right]$$

It is then easy to differentiate and get

$$E[X] = -jM_X^{\prime}(jv)|_{v=0} = m_X$$

and

$$E\left[X^2\right] = (-j)^2 M_X^{\prime\prime}(jv)|_{v=0} = \sigma_X^2 + m_X^2$$

from which it follows that the variance is σ_X^2.

Problem 4.36
(a) $K = 1/\pi$;
(b) $E[X]$ is not defined, but one could argue that it is zero from the oddness of the integrand for this expectation. If one writes down the integral for the second moment, it clearly does not converge (the integrand approaches a constant as $|x| \to \infty$).
(c) The answer is given;
(d) Compare the form of the characteristic function with the answer given in (c) and do a suitable redefinition of variables.

Problem 4.37
(a) The characteristic function is found from

$$M_Y(jv) = E\left\{e^{jvY}\right\} = E\left\{e^{jv\sum_{i=1}^{n} X_i^2}\right\} = E\left\{\prod_{i=1}^{n} e^{jvX_i^2}\right\}$$

But

$$\begin{aligned} E\left\{e^{jvX_i^2}\right\} &= \int_{-\infty}^{\infty} e^{jvx^2} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-\left(1/2\sigma^2 - jv\right)x^2}}{\sqrt{2\pi\sigma^2}} dx \\ &= \left(1 - j2v\sigma^2\right)^{-1/2} \end{aligned}$$

which follows by using the definite integral defined at the beginning of Problem 4.35. Thus

$$M_Y(jv) = \left(1 - j2v\sigma^2\right)^{-N/2}$$

(b) The given pdf follows easily by letting $\alpha = 2\sigma^2$ in the given Fourier transform pair.

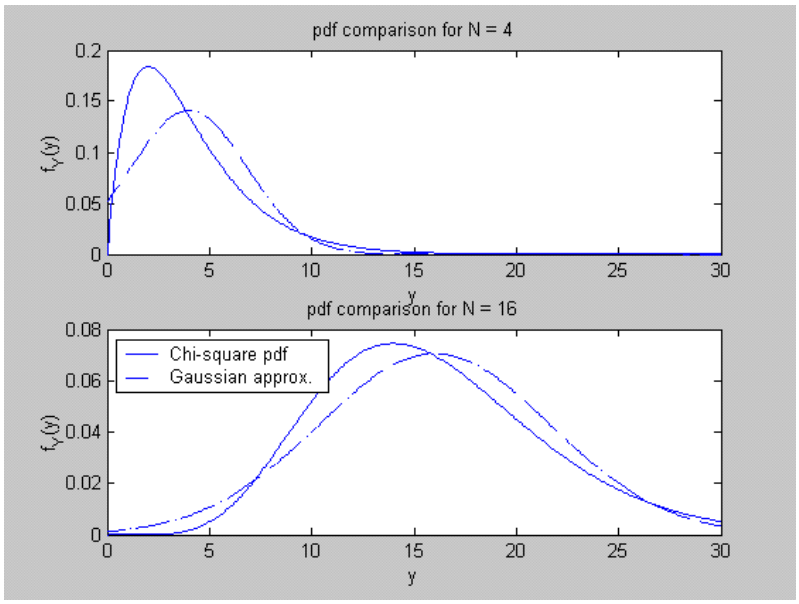

Figure 4.1:

(c) The mean of Y is

$$E[Y] = N\sigma^2$$

by taking the sum of the expectations of the separate terms of the sum defining Y. The mean-square value of X_i^2 is

$$E\left[\left(X_i^2\right)^2\right] = (-j)^2 \left.\frac{d^2\left[\left(1 - j2v\sigma^2\right)^{-1/2}\right]}{dv^2}\right|_{v=0} = 3\sigma^4$$

Thus,

$$\text{var}\left[\left(X_i^2\right)^2\right] = 3\sigma^4 - \sigma^4 = 2\sigma^4$$

Since the terms in the sum defining Y are independent, $\text{var}[Y] = 2N\sigma^4$. The mean and variance of Y can be put into the definition of a Gaussian pdf to get the desired approximation.

(d) A comparison of the cases $N = 4$ and $N = 16$ is shown in Figure 4.1.

Note the extreme difference between exact and approximation due to the central limit theorem not being valid for the former case.

(e) Evident by direct substitution.

Problem 4.38
This is a matter of plotting $Q(x) = \int_x^\infty \frac{\exp\left(-t^2/2\right)}{\sqrt{2\pi}} dx$ and $Q_a(x) = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}x}$ on the same set of axes.

Problem 4.39
By definition, the cdf of a Gaussian random variable is

$$F_X(x) = \int_{-\infty}^{x} \frac{e^{-\frac{(u-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} du = 1 - \int_{x}^{\infty} \frac{e^{-\frac{(u-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} du$$

Change variables to

$$v = \frac{u-m}{\sigma}$$

with the result that

$$F_X(x) = 1 - \int_{\frac{x-m}{\sigma}}^{x} \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} du = 1 - Q\left(\frac{x-m}{\sigma}\right)$$

A plot may easily be obtained by a MATLAB program. It is suggested that you use the MATLAB function $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt$.

Problem 4.40
The exact probability is

$$P(|X| \le k\sigma_X) = P(-k\sigma_X \le X \le k\sigma_X) = \begin{cases} k\sigma_x, \; k\sigma_x \le 1 \\ 1, \; k\sigma_x > 1 \end{cases}$$

Chebyshev's bound is

$$P(|X| \le k\sigma_X) \ge 1 - \frac{1}{k^2}, \; k > 0$$

A plot is left to the student.

Problem 4.41
(a) The probability is

$$\begin{aligned} P(X \le 11) &= \int_{-\infty}^{11} \frac{e^{-(x-12)^2/30}}{\sqrt{30\pi}} dx = Q\left(1/\sqrt{15}\right) \\ &= \frac{1}{2}\text{erfc}\left(1/\sqrt{30}\right) = 0.398 \end{aligned}$$

(b) This probability is

$$\begin{aligned} P(10 < X \leq 12) &= \int_{10}^{12} \frac{e^{-(x-12)^{2/30}}}{\sqrt{30\pi}} dx = \frac{1}{2} - Q\left(2/\sqrt{15}\right) \\ &= \frac{1}{2}\text{erf}\left(2/\sqrt{30}\right) = 0.197 \end{aligned}$$

(c) The desired probability is

$$\begin{aligned} P(11 < X \leq 13) &= \int_{11}^{13} \frac{e^{-(x-12)^{2/30}}}{\sqrt{30\pi}} dx = 1 - 2Q\left(1/\sqrt{15}\right) \\ &= \text{erf}\left(1/\sqrt{30}\right) = 0.204 \end{aligned}$$

(d) The result is

$$\begin{aligned} P(9 < X \leq 12) &= \int_{9}^{12} \frac{e^{-(x-12)^{2/30}}}{\sqrt{30\pi}} dx = \frac{1}{2} - Q\left(3/\sqrt{15}\right) \\ &= \frac{1}{2}\text{erf}\left(3/\sqrt{30}\right) = 0.281 \end{aligned}$$

Problem 4.42
Observe that

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, dx \\ &\geq \int_{|x-m|>k\sigma} (x-m)^2 f_X(x)\, dx \\ &\geq \int_{|x-m|>k\sigma} k^2\sigma^2 f_X(x)\, dx \\ &= k^2\sigma^2 P\{|X-m| > k\sigma\} = k^2\sigma^2 \{1 - P[|X-m| > k\sigma]\} \end{aligned}$$

or

$$P[|X-m| < k\sigma] \geq 1 - \frac{1}{k^2}$$

Problem 4.43
The mean is 0 by even symmetry of the pdf. Thus, the variance is

$$\begin{aligned}
\text{var } [X] &= E\left[X^2\right] \\
&= \int_{-\infty}^{\infty} x^2 \left(\frac{a}{2}\right) \exp\left(-a1 \times 1\right) dx \\
&= 2\left(\frac{a}{2}\right) \int_{0}^{\infty} x^2 \exp\left(-ax\right) dx \\
&= 2a \int_{0}^{\infty} \left(\frac{y}{a}\right)^2 \exp\left(-y\right) \frac{dy}{a} \\
&= 2a \left(\frac{1}{a^3}\right) \left(\frac{1}{2}\right) = 1/a^2
\end{aligned}$$

4.2 Computer Exercises

Computer Exercise 4.1

```
% ce4_1.m: Generate an exponentially distributed random variable
% with parameter a from a uniform random variable in [0, 1)
%
clf
a = input('Enter the parameter of the exponential pdf: f(x) = a*exp(-a*x)*u(x): ');
N = input('Enter number of exponential random variables to be generated: ');
U = rand(1,N);
V = -(1/a)*log(U);
[M, X] = hist(V);
disp(' ')
disp('No. of random variables generated')
disp(N)
disp(' ')
disp('Bin centers')
disp(X)
disp(' ')
disp('No. of random variable counts in each bin')
disp(M)
disp(' ')
norm_hist = M/(N*(X(2)-X(1)));
plot(X, norm_hist, 'o')
hold
plot(X, a*exp(-a*X), '--'), xlabel('x'), ylabel('f_X(x)'),...
```

```
title([‘Theoretical pdf and histogram for ’,num2str(N),
‘ computer generated exponential random variables’])
legend([‘Histogram points’],[‘Exponential pdf; a = ’, num2str(a)])
```

A typical run follows:

```
>> ce4_1
Enter the parameter of the exponential pdf: f(x) = a*exp(-a*x)*u(x): 2
Enter number of exponential random variables to be generated: 5000
No. of random variables generated
5000
Bin centers
Columns 1 through 7
0.1876 0.5626 0.9375 1.3125 1.6875 2.0625 2.4375
Columns 8 through 10
2.8125 3.1875 3.5624
No. of random variable counts in each bin
Columns 1 through 6
2677 1212 590 263 129 71
Columns 7 through 10
34 9 11 4
Current plot held
```

Computer Exercise 4.2

```
% ce4_2.m: Generate pairs of Gauss. random variables from Rayleigh & uniform RVs
%
clf
m = input(‘Enter the mean of the Gaussian random variables: ’);
sigma = input(‘Enter the standard deviation of the Gaussian random variables: ’);
N = input(‘Enter number of Gaussian random variable pairs to be generated: ’);
U = rand(1,N);
V = rand(1,N);
R = sqrt(-2*log(V));
X = sigma*R.*cos(2*pi*U)+m;
Y = sigma*R.*sin(2*pi*U)+m;
disp(‘ ’)
disp(‘Covarance matrix of X and Y vectors:’)
disp(cov(X,Y))
disp(‘ ’)
[MX, X_bin] = hist(X, 20);
norm_MX = MX/(N*(X_bin(2)-X_bin(1)));
[MY, Y_bin] = hist(Y, 20);
```

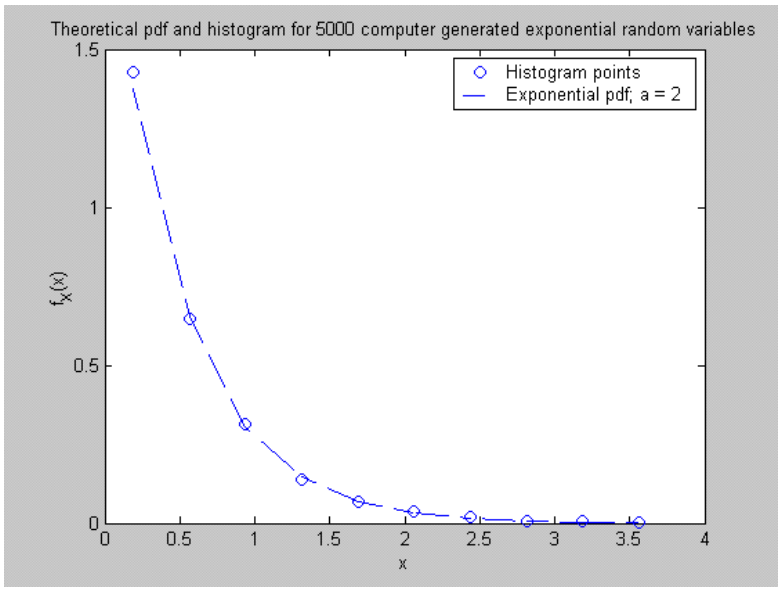

Figure 4.2:

```
norm_MY = MY/(N*(Y_bin(2)-Y_bin(1)));
gauss_pdf_X = exp(-(X_bin - m).^2/(2*sigma^2))/sqrt(2*pi*sigma^2);
gauss_pdf_Y = exp(-(Y_bin - m).^2/(2*sigma^2))/sqrt(2*pi*sigma^2);
subplot(2,1,1), plot(X_bin, norm_MX, 'o')
hold
subplot(2,1,1), plot(X_bin, gauss_pdf_X, '–'), xlabel('x'), ylabel('f_X(x)'),...
title(['Theoretical pdfs and histograms for ',num2str(N),
' computer generated independent Gaussian RV pairs'])
legend(['Histogram points'],['Gauss pdf; \sigma, \mu = ', num2str(sigma),', ', num2str(m)])
subplot(2,1,2), plot(Y_bin, norm_MY, 'o')
hold
subplot(2,1,2), plot(Y_bin, gauss_pdf_Y, '–'), xlabel('y'), ylabel('f_Y(y)')
```

A typical run follows:

```
>> ce4_2
Enter the mean of the Gaussian random variables: 1
Enter the standard deviation of the Gaussian random variables: 3
Enter number of Gaussian random variable pairs to be generated: 2000
Covarance matrix of X and Y vectors:
8.5184      0.0828
```

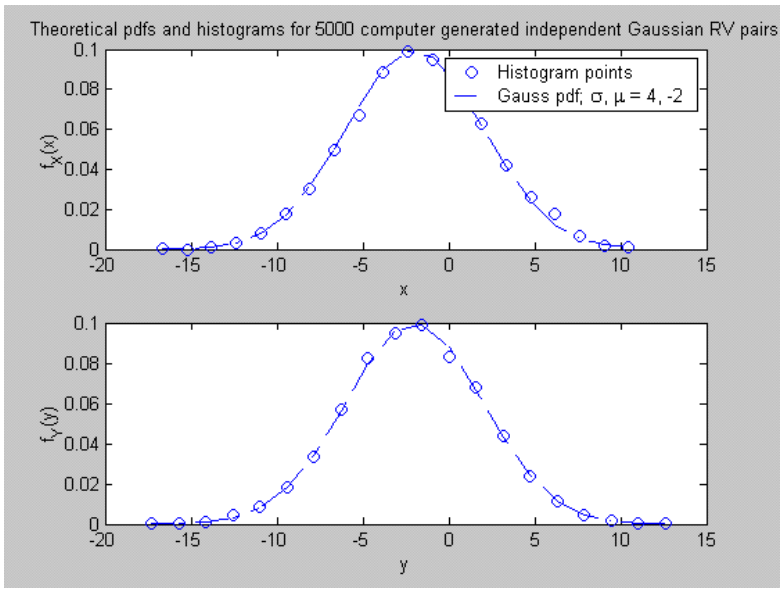

Figure 4.3:

```
0.0828      8.9416
Current plot held
Current plot held
```

Computer Exercise 4.3

```
% ce4_3.m: Generate a sequence of Gaussian random variables with specified correlation
% between adjacent RVs
%
clf
m = input('Enter the mean of the Gaussian random variables: ');
sigma = input('Enter the standard deviation of the Gaussian random variables: ');
rho = input('Enter correlation coefficient between adjacent Gauss. RVs: ');
N = input('Enter number of Gaussian random variables to be generated: ');
var12 = sigma^2*(1 - rho^2);
X = [];
X(1) = sigma*randn(1)+m;
for k = 2:N
m12 = m + rho*(X(k-1) - m);
X(k) = sqrt(var12)*randn(1) + m12;
end
```

```
[MX, X_bin] = hist(X, 20);
norm_MX = MX/(N*(X_bin(2)-X_bin(1)));
gauss_pdf_X = exp(-(X_bin - m).^2/(2*sigma^2))/sqrt(2*pi*sigma^2);
subplot(2,1,1), plot(X_bin, norm_MX, 'o')
hold
subplot(2,1,1), plot(X_bin, gauss_pdf_X, '-'), xlabel('x'), ylabel('f_X(x)'),...
title(['Theoretical pdf and histogram for ',num2str(N),
' computer generated Gaussian RVs'])
legend(['Histogram points'],['Gauss pdf; \sigma,
      \mu = ', num2str(sigma),', ', num2str(m)],2)
Z = X(1:50);
subplot(2,1,2), plot(Z, 'x'), xlabel('k'), ylabel('X(k)')
title(['Gaussian sample values with correlation ', num2str(rho), ' between samples'])
```

A typical run follows:

```
>> ce4_3
Enter the mean of the Gaussian random variables: 0
Enter the standard deviation of the Gaussian random variables: 2
Enter correlation coefficient between adjacent Gaussian random variables: .7
Enter number of Gaussian random variables to be generated: 2000
Current plot held
```

Computer Exercise 4.4

```
% ce4_4.m: Testing the validity of the Central Limit Theorem with sums of uniform
% random numbers
%
clf
N = input('Enter number of Gaussian random variables to be generated: ');
M = input('Enter number of uniform RVs to be summed to generate one GRV: ');
% Mean of uniform = 0; variance = 1/12
Y = rand(M,N)-0.5;
X_gauss = sum(Y)/(sqrt(M/12));
disp(' ')
disp('Estimated variance of the Gaussian RVs:')
disp(cov(X_gauss))
disp(' ')
[MX, X_bin] = hist(X_gauss, 20);
norm_MX = MX/(N*(X_bin(2)-X_bin(1)));
gauss_pdf_X = exp(-X_bin.^2/2)/sqrt(2*pi);
plot(X_bin, norm_MX, 'o')
hold
```

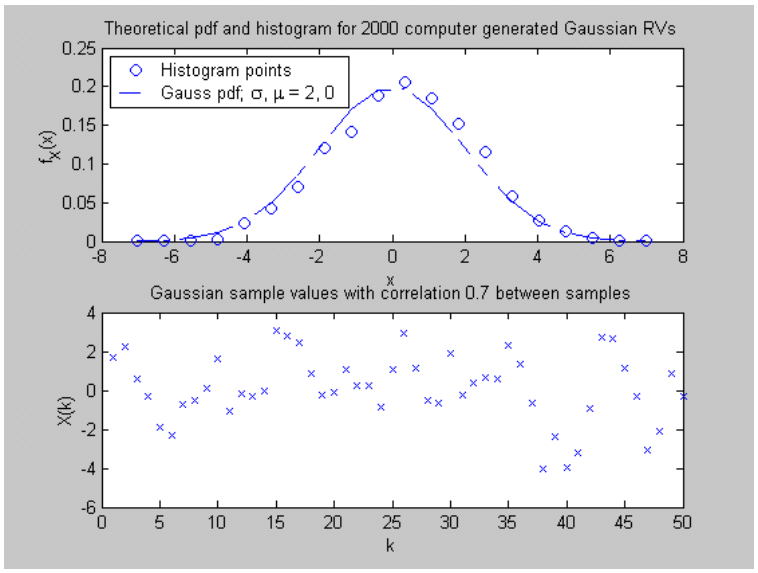

Figure 4.4:

```
plot(X_bin, gauss_pdf_X, '–'), xlabel('x'), ylabel('f_X(x)'),...
title(['Theoretical pdf & histogram for ',num2str(N),
' Gaussian RVs each generated as the sum of ', num2str(M), ' uniform RVs'])
legend(['Histogram points'], ['Gauss pdf; \sigma = 1, \mu = 0'],2)
```

A typical run follows:

```
>> ce4_4
Enter number of Gaussian random variables to be generated: 10000
Enter number of uniform random variables to be summed to generate one GRV: 30
Estimated variance of the Gaussian RVs:
1.0075
Current plot held
```

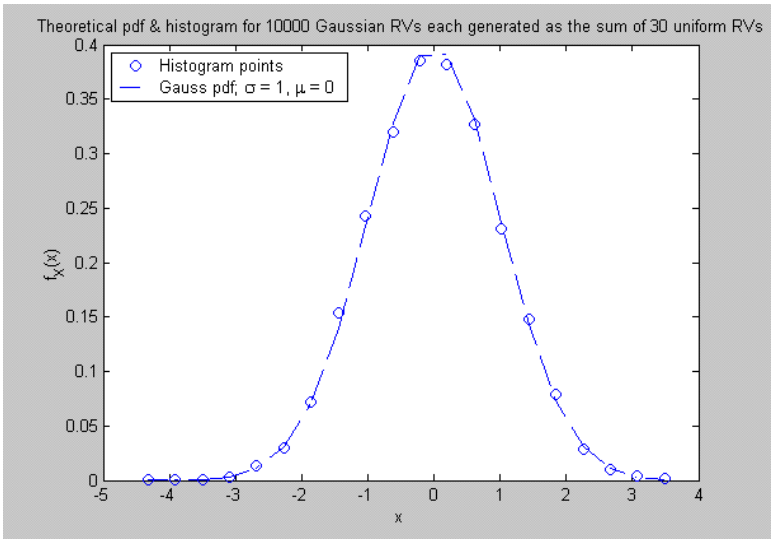
Theoretical pdf & histogram for 10000 Gaussian RVs each generated as the sum of 30 uniform RVs
Histogram points
Gauss pdf; σ = 1, μ = 0
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
x
f_X(x)

Figure 4.5:

Chapter 5

Random Signals and Noise

5.1 Problem Solutions

Problem 5.1
The various sample functions are as follows. Sample functions for case (a) are horizontal lines at levels $A,\ 0,\ -A$, each case of which occurs equally often (with probability 1/3). Sample functions for case (b) are horizontal lines at levels $5A,\ 3A,\ A,\ -A,\ -3A,\ -5A$, each case of which occurs equally often (with probability 1/6). Sample functions for case (c) are horizontal lines at levels $4A,\ 2A,\ -2A,\ -4A$, or oblique straight lines of slope A or $-A$, each case of which occurs equally often (with probability 1/6).

Problem 5.2

a. For case (a) of problem 5.1, since the sample functions are constant with time and are less than or equal 2A, the probability is one. For case (b) of problem 5.1, since the sample functions are constant with time and 4 out of 6 are less than or equal to 2A, the probability is 2/3. For case (c) of problem 5.1, the probability is again 2/3 because 4 out of 6 of the sample functions will be less than $2A$ at $t = 4$.

b. The probabilities are now 2/3, 1/2, and 1/2, respectively.

c. The probabilities are 1, 2/3, and 5/6, respectively.

Problem 5.3

a. The sketches would consist of squarewaves of random delay with respect to $t = 0$.

b. $X(t)$ takes on only two values A and $-A$, and these are equally likely. Thus

$$f_X(x) = \frac{1}{2}\delta(x - A) + \frac{1}{2}\delta(x + A)$$

Problem 5.4

a. The sample functions at the integrator output are of the form

$$Y^{(i)}(t) = \int^t A\cos(\omega_0\lambda)\,d\lambda = \frac{A}{\omega_0}\sin\omega_0 t$$

where A is Gaussian.

b. $Y^{(i)}(t_0)$ is Gaussian with mean zero and standard deviation

$$\sigma_Y = \sigma_a \frac{|\sin(\omega_0 t_0)|}{\omega_0}$$

c. Not stationary and not ergodic.

Problem 5.5

a. By inspection of the sample functions, the mean is zero. Considering the time average correlation function, defined as

$$R(\lambda) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\lambda)\,dt = \frac{1}{T_0}\int_{T_0} x(t)\,x(t+\lambda)\,dt$$

where the last expression follows because of the periodicity of $x(t)$, it follows from a sketch of the integrand that

$$R(\tau) = A^2(1 - 4\lambda/T_0), \quad 0 \le \lambda \le T_0/2$$

Since $R(\tau)$ is even and periodic, this defines the autocorrelation function for all λ.

b. Yes, it is wide sense stationary. The random delay, λ, being over a full period, means that no untypical sample functions occur.

Problem 5.6

a. The time average mean is zero. The statistical average mean is

$$\begin{aligned}
E\left[X\left(t\right)\right] &= \int_{\pi/4}^{\pi/2} A\cos\left(2\pi f_0 t+\theta\right)\frac{d\theta}{\pi/4} = \frac{4A}{\pi}\sin\left(2\pi f_0 t+\theta\right)\Big|_{\pi/4}^{\pi/2} \\
&= \frac{4A}{\pi}\left[\sin\left(2\pi f_0 t+\pi/2\right)-\sin\left(2\pi f_0 t+\pi/4\right)\right] \\
&= \frac{4A}{\pi}\left[\left(1-\frac{1}{\sqrt{2}}\right)\cos\left(2\pi f_0 t\right)-\frac{1}{\sqrt{2}}\sin\left(2\pi f_0 t\right)\right]
\end{aligned}$$

The time average variance is $A^2/2$. The statistical average second moment is

$$\begin{aligned}
E\left[X^2\left(t\right)\right] &= \int_{\pi/4}^{\pi/2} A^2\cos^2\left(2\pi f_0 t+\theta\right)\frac{d\theta}{\pi/4} \\
&= \frac{2A^2}{\pi}\left[\int_{\pi/4}^{\pi/2} d\theta+\int_{\pi/4}^{\pi/2}\cos\left(4\pi f_0 t+2\theta\right)d\theta\right] = \frac{2A^2}{\pi}\left[\frac{\pi}{4}+\frac{1}{2}\sin\left(4\pi f_0 t+2\theta\right)\Big|_{\pi/4}^{\pi/2}\right] \\
&= \frac{A^2}{2}+\frac{A^2}{\pi}\left[\sin\left(4\pi f_0 t+\pi\right)-\sin\left(4\pi f_0 t+\frac{\pi}{2}\right)\right] \\
&= \frac{A^2}{2}-\frac{A^2}{\pi}\left[\sin\left(4\pi f_0 t\right)+\cos\left(4\pi f_0 t\right)\right]
\end{aligned}$$

The statistical average variance is this expression minus $E^2\left[X\left(t\right)\right]$.

b. The time average autocorrelation function is

$$\left\langle x\left(t\right)x\left(t+\tau\right)\right\rangle = \frac{A^2}{2}\cos\left(2\pi f_0\tau\right)$$

The statistical average autocorrelation function is

$$\begin{aligned}
R\left(\tau\right) &= \int_{\pi/4}^{\pi/2} A^2\cos\left(2\pi f_0 t+\theta\right)\cos\left(2\pi f_0\left(t+\tau\right)+\theta\right)\frac{d\theta}{\pi/4} \\
&= \frac{2A^2}{\pi}\int_{\pi/4}^{\pi/2}\left[\cos\left(2\pi f_0\tau\right)+\cos\left(2\pi f_0\left(2t+\tau\right)+2\theta\right)\right]d\theta \\
&= \frac{A^2}{2}\cos\left(2\pi f_0\tau\right)+\frac{A^2}{\pi}\sin\left(2\pi f_0\left(2t+\tau\right)+2\theta\right)\Big|_{\pi/4}^{\pi/2} \\
&= \frac{A^2}{2}\cos\left(2\pi f_0\tau\right)+\frac{A^2}{\pi}\left[\sin\left(2\pi f_0\left(2t+\tau\right)+\pi\right)-\sin\left(2\pi f_0\left(2t+\tau\right)+\frac{\pi}{2}\right)\right] \\
&= \frac{A^2}{2}\cos\left(2\pi f_0\tau\right)-\frac{A^2}{\pi}\left[\sin 2\pi f_0\left(2t+\tau\right)+\cos 2\pi f_0\left(2t+\tau\right)\right]
\end{aligned}$$

c. No. The random phase, not being uniform over $(0, 2\pi)$, means that for a given time, t, certain phases will be favored over others.

Problem 5.7

a. The mean is zero. The mean-square value is

$$\begin{aligned} E\left[Z^2\right] &= E\left\{\left[X(t)\cos(\omega_0 t)\right]^2\right\} \\ &= E\left[X^2(t)\right]\cos^2(\omega_0 t) = \sigma_X^2 \cos^2(\omega_0 t) \end{aligned}$$

The process is not stationary since its second moment depends on time.

b. Again, the mean is clearly zero. The second moment is

$$\begin{aligned} E\left[Z^2\right] &= E\left\{\left[X(t)\cos(\omega_0 t + \theta)\right]^2\right\} \\ &= E\left[X^2(t)\right] E\left[\cos^2(\omega_0 t + \theta)\right] = \frac{\sigma_X^2}{2} \end{aligned}$$

Problem 5.8

The pdf of the noise voltage is

$$f_X(x) = \frac{e^{-(x-2)^2/10}}{\sqrt{10\pi}}$$

Problem 5.9

a. Suitable;

b. Suitable;

c. Not suitable because the Fourier transform is not everywhere nonnegative;

d. Not suitable for the same reason as (c);

e. Not suitable because it is not even.

Problem 5.10
Use the Fourier transform pair

$$2W\text{sinc}\left(2W\tau\right) \leftrightarrow \Pi\left(f/2W\right)$$

with $W = 2 \times 10^6$ Hz to get the autocorrelation function as

$$\begin{aligned} R_X\left(\tau\right) &= N_0 W \,\text{sinc}\left(2W\tau\right) = \left(2 \times 10^{-8}\right)\left(2 \times 10^6\right)\text{sinc}\left(4 \times 10^6 \tau\right) \\ &= 0.02\,\text{sinc}\left(4 \times 10^6 \tau\right) \end{aligned}$$

Problem 5.11

a. The $r\left(\tau\right)$ function, (5.55), is

$$\begin{aligned} r\left(\tau\right) &= \frac{1}{T}\int_{-\infty}^{\infty} p\left(t\right) p\left(t+\tau\right) dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2-\tau} \cos\left(\pi t/T\right)\cos\left[\frac{\pi\left(t+\tau\right)}{T}\right] dt, \quad 0 \le \tau \le T/2 \\ &= \frac{1}{2}\left(1-\tau/T\right)\cos\left(\frac{\pi\tau}{T}\right), \quad 0 \le \tau \le T/2 \end{aligned}$$

Now $r\left(\tau\right) = r\left(-\tau\right)$ and $r\left(\tau\right) = 0$ for $|\tau| > T/2$. Therefore it can be written as

$$r\left(\tau\right) = \frac{1}{2}\Lambda\left(\tau/T\right)\cos\left(\frac{\pi\tau}{T}\right)$$

Hence, by (5.54)

$$R_a\left(\tau\right) = \frac{A^2}{2}\Lambda\left(\tau/T\right)\cos\left(\frac{\pi\tau}{T}\right)$$

The power spectral density is

$$S_a\left(f\right) = \frac{A^2 T}{4}\left\{\text{sin c}^2\left[T\left(f-\frac{1}{2T}\right)\right] + \text{sinc}^2\left[T\left(f+\frac{1}{2T}\right)\right]\right\}$$

b. The autocorrelation function now becomes

$$R_b\left(\tau\right) = A^2\left[2r\left(\tau\right) + r\left(\tau+T\right) + r\left(\tau-T\right)\right]$$

which is (5.62) with $g_0 = g_1 = 1$. The corresponding power spectrum is

$$S_b\left(f\right) = A^2 T\left[\text{sinc}^2\left(Tf-0.5\right) + \text{sin c}^2\left(Tf+0.5\right)\right]\cos^2\left(\pi f T\right)$$

c. The plots are left for the student.

Problem 5.12
The cross-correlation function is

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = R_n(\tau) + E\left\{A^2\cos(\omega_0 t+\Theta)\times\sin[\omega_0(t+\tau)+\Theta]\right\} \\ &= B\Lambda(\tau/\tau_0) + \frac{A^2}{2}\sin(\omega_0\tau) \end{aligned}$$

Problem 5.13

a. By definition

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] \\ &= E[X(t)X(t+\tau)Y(t)Y(t+\tau)] \\ &= E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)] \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

b. Since a product in the time domain is convolution in the frequency domain, it follows that

$$S_Z(f) = S_X(f) * S_Y(f)$$

c. Use the transform pairs

$$2W\operatorname{sinc}(2W\tau) \longleftrightarrow \Pi(f/2W)$$

and

$$\cos(2\pi f_0\tau) \longleftrightarrow \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

Also,

$$R_Y(\tau) = E\{4\cos(50\pi t+\theta)\cos[50\pi(t+\tau)+\theta]\} = 2\cos(50\pi\tau)$$

Using the first transform pair, we have

$$R_Y(\tau) = 500\operatorname{sinc}(100\tau)$$

This gives

$$\begin{aligned} R_Z(\tau) &= [500\,\mathrm{sinc}\,(100\tau)]\,[2\cos(50\pi\tau)] \\ &= 1000\,\mathrm{sinc}\,(100\tau)\cos(50\pi\tau) \end{aligned}$$

and

$$S_Z(f) = 5\times 10^3\left[\prod\left(\frac{f-25}{200}\right)+\prod\left(\frac{f+25}{200}\right)\right]$$

Problem 5.14

a. $E\left[X^2(t)\right] = R(0) = 5$ W; $E^2[X(t)] = \lim_{\tau\to\infty} R(\tau) = 4$ W;

$\sigma_X^2 = E\left[X^2(t)\right] - E^2[X(t)] = 5-4 = 1$ W.

b. dc power $= E^2[X(t)] = 4$ W.

c. Total power $= E\left[X^2(t)\right] = 5$ W.

d. $S(f) = 4\delta(f) + 5\,\mathrm{sinc}^2(5f)$.

Problem 5.15

a. The autocorrelation function is

$$\begin{aligned} R_X(\tau) &= E[Y(t)\,Y(t+\tau)] \\ &= E\{[X(t)+X(t-T)][X(t+\tau)+X(t+\tau-T)]\} \\ &= E[X(t)\,X(t+\tau)] + E[X(t)\,X(t+\tau+\tau)] \\ &\quad +E[X(t-T)\,X(t+\tau)] + E[X(t-T)\,X(t+\tau-T)] \\ &= 2R_X(\tau) + R_X(\tau-T) + R_X(\tau+T) \end{aligned}$$

b. Application of the time delay theorem of Fourier transforms gives

$$\begin{aligned} S_Y(f) &= 2S_X(f) + S_X(f)\,[\exp(-j2\pi fT) + \exp(j2\pi fT)] \\ &= 2S_X(f) + 2S_X(f)\cos(2\pi fT) \\ &= 4S_X(f)\cos^2(\pi fT) \end{aligned}$$

c. Use the transform pair

$$R_X(\tau) = 6\Lambda(\tau) \longleftrightarrow S_X(f) = 6\,\mathrm{sinc}^2(f)$$

and the result of (b) to get

$$S_Y(f) = 24\,\mathrm{sinc}^2(f)\cos^2(\pi f/4)$$

Problem 5.16

a. The student should carry out the sketch.

b. dc power $= \int_{0-}^{0+} S(f)\,df = 5$ W.

c. Total power $= \int_{-\infty}^{\infty} S(f)\,df = 5 + 10/5 = 7$ W

Problem 5.17

a. This is left for the student.

b. The dc power is either $\lim_{\tau\to\infty} R_X(\tau)$ or $\int_{0-}^{0+} S_X(f)\,df$. Thus (1) has 0 W dc power, (2) has dc power K_1 W, and (3) has dc power K_2 W.

c. Total power is given by $R_X(0)$ or the integral of $\int_{-\infty}^{\infty} S_X(f)\,df$ over all frequency. Thus (1) has total power K, (2) has total power $2K_1 + K_2$, and (3) has total power

$$P_3 = K\int_{-\infty}^{\infty} e^{-\alpha f^2}df + K_1 + K_1 + K_2 = \sqrt{\frac{\pi}{\alpha}}K + 2K_1 + K_2$$

where the integral is evaluated by means of a table of definite integrals.

d. (2) has a periodic component of frequency b Hz, and (3) has a periodic component of 10 Hz, which is manifested by the two delta functions at $f = \pm 10$ Hz present in the power spectrum.

Problem 5.18

a. The output power spectral density is

$$S_{\text{out}}(f) = 10^{-5}\Pi\left(f/2\times 10^6\right)$$

Note that the rectangular pulse function does not need to be squared because its amplitude squared is unity.

b. The autocorrelation function of the output is

$$R_{\text{out}}(\tau) = 2 \times 10^6 \times 10^{-5} \text{sinc}\left(2 \times 10^6 \tau\right) = 20 \text{ sinc}\left(2 \times 10^6 \tau\right)$$

c. The output power is 20 W, which can be found from $R_{\text{out}}(0)$ or the integral of $S_{\text{out}}(f)$.

Problem 5.19

a. By assuming a unit impulse at the input, the impulse response is

$$h(t) = \frac{1}{T}\left[u(t) - u(t-T)\right]$$

b. Use the time delay theorem of Fourier transforms and the Fourier transform of a rectangular pulse to get

$$H(f) = \text{sinc}(fT)\, e^{-j\pi f T}$$

c. The output power spectral density is

$$S_0(f) = \frac{N_0}{2}\text{sinc}^2(fT)$$

d. Use $F\left[\Lambda(\tau/\tau_0)\right] = \tau_0 \text{sinc}^2(\tau_0 f)$ to get

$$R_0(\tau) = \frac{N_0}{2T}\Lambda(\tau/T)$$

e. By definition of the equivalent noise bandwidth

$$E\left[Y^2\right] = H_{\max}^2 B_N N_0$$

The output power is also $R_0(0) = N_0/2T$. Equating the two results and noting that $H_{\max} = 1$, gives $B_N = 1/2T$ Hz.

f. Evaluate the integral of the power spectral density over all frequency:

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} \frac{N_0}{2}\text{sinc}^2(fT)\, df = \frac{N_0}{2T}\int_{-\infty}^{\infty} \text{sinc}^2(u)\, du = \frac{N_0}{2T} = R_0(0)$$

Problem 5.20

a. The output power spectral density is

$$S_0(f) = \frac{N_0/2}{1 + (f/f_3)^4}$$

b. The autocorrelation function of the output, by inverse Fourier transforming the output power spectral density, is

$$\begin{aligned} R_0(\tau) &= F^{-1}[S_0(f)] = \int_{-\infty}^{\infty} \frac{N_0/2}{1 + (f/f_3)^4} e^{-j2\pi f\tau} df \\ &= N_0 \int_0^{\infty} \frac{\cos(2\pi f\tau)}{1 + (f/f_3)^4}\, df = N_0 f_3 \int_0^{\infty} \frac{\cos(2\pi f_3 \tau x)}{1 + x^4}\, dx \\ &= \frac{\pi f_3 N_0}{2} e^{-\sqrt{2}\pi f_3 \tau} \cos\left(\sqrt{2}\pi f_3 \tau - \pi/4\right) \end{aligned}$$

c. Yes. $R_0(0) = \frac{\pi f_3 N_0}{2} = N_0 B_N, \quad B_N = \pi f_3/2$

Problem 5.21
We have

$$|H(f)|^2 = \frac{(2\pi f)^2}{(2\pi f)^4 + 5,000}$$

Thus

$$|H(f)| = \frac{|2\pi f|}{\sqrt{(2\pi f)^4 + 5,000}}$$

This could be realized with a second-order Butterworth filter in cascade with a differentiator.

Problem 5.22

a. The power spectral density is

$$S_Y(f) = \frac{N_0}{2} \Pi(f/2B)$$

The autocorrelation function is

$$R_Y(\tau) = N_0 B \operatorname{sinc}(2B\tau)$$

b. The transfer function is

$$H(f) = \frac{A}{\alpha + j2\pi f}$$

The power spectral density and autocorrelation function are, respectively,

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) = \frac{A^2}{\alpha^2 + (2\pi f)^2} \frac{B}{1 + (2\pi\beta f)^2} \\ &= \frac{A^2/\alpha^2}{1 + (2\pi f/\alpha)^2} \frac{B}{1 + (2\pi\beta f)^2} = \frac{A^2 B/\alpha^2}{1 - \alpha^2\beta^2} \left[\frac{1}{1 + (2\pi f/\alpha)^2} - \frac{\alpha^2\beta^2}{1 + (2\pi\beta f)^2} \right] \end{aligned}$$

Use the transform pair $\exp(-|\tau|/\tau_0) \longleftrightarrow \frac{2\tau_0}{1+(2\pi f\tau_0)^2}$ to find that the corresponding autocorrelation function is

$$R_Y(t) = \frac{A^2 B/\alpha}{2\left[1 - (\alpha\beta)^2\right]} \left[e^{-\alpha|\tau|} - \alpha\beta\, e^{-|\tau|/\beta} \right] \quad , \quad \alpha \neq 1/\beta$$

Problem 5.23

a. $E[Y(t)] = 0$ because the mean of the input is zero.

b. The output power spectrum is

$$S_Y(f) = \frac{1}{(10\pi)^2} \frac{1}{1 + (f/5)^2}$$

which is obtained by Fourier transforming the impulse response to get the transfer function and applying (5.90).

c. Use the transform pair $\exp(-|\tau|/\tau_0) \longleftrightarrow \frac{2\tau_0}{1+(2\pi f\tau_0)^2}$ to find the power spectrum as

$$R_Y(\tau) = \frac{1}{10\pi} e^{-10\pi|\tau|}$$

d. Since the input is Gaussian, so is the output. Also, $E[Y] = 0$ and $\mathrm{var}[Y] = R_Y(0) = 1/(10\pi)$, so

$$f_Y(y) = \frac{e^{-y^2/2\sigma_Y^2}}{\sqrt{2\pi\sigma_Y^2}}$$

where $\sigma_Y^2 = \frac{1}{10\pi}$.

e. Use (4.188) with

$$x = y_1 = Y(t_1),\ y = y_2 = Y(t_2),\ m_x = m_y = 0, \text{ and } \sigma_x^2 = \sigma_y^2 = 1/10\pi$$

Also

$$\rho(\tau) = \frac{R_Y(\tau)}{R_Y(0)} = e^{-10\pi|\tau|}$$

Set $\tau = 0.03$ to get $\rho(0.03) = 0.3897$. Put these values into (4.188).

Problem 5.24
Use the integrals

$$I_1 = \int_{-\infty}^{\infty} \frac{b_0 ds}{(a_0 s + a_1)(-a_0 s + a_1)} = \frac{j\pi b_0}{a_0 a_1}$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{\left(b_0 s^2 + b_1\right) ds}{(a_0 s^2 + a_1 s + a_2)(a_0 s^2 - a_1 s + a_2)} = j\pi \frac{-b_0 + a_0 b_1/a_2}{a_0 a_1}$$

to get the following results for B_N.

Filter Type	**First Order**	**Second Order**
Chebyshev	$\frac{\pi f_c}{2\epsilon}$	$\frac{\pi f_c/\epsilon^2}{\sqrt{1+1/\epsilon^2}\sqrt{\sqrt{1+1/\epsilon^2}-1}}$
Butterworth	$\frac{\pi}{2} f_c$	$\frac{\pi}{6} f_c$

Problem 5.25
The transfer function is

$$\begin{aligned} H(s) &= \frac{\omega_c^2}{\left(s + \omega_c/\sqrt{2} + j\omega_c/\sqrt{2}\right)\left(s + \omega_c/\sqrt{2} - j\omega_c/\sqrt{2}\right)} \\ &= \frac{A}{\left(s + \omega_c/\sqrt{2} + j\omega_c/\sqrt{2}\right)} + \frac{A^*}{\left(s + \omega_c/\sqrt{2} - j\omega_c/\sqrt{2}\right)} \end{aligned}$$

where $A = j\omega_c/2^{1/2}$. This is inverse Fourier transformed, with $s = j\omega$, to yield

$$h(t) = \sqrt{2}\omega_c e^{-\omega_c t/\sqrt{2}} \sin\left(\frac{\omega_c t}{\sqrt{2}}\right) u(t)$$

for the impulse response. Now

$$\frac{1}{2}\int_{-\infty}^{\infty}|h(t)|^2\,dt = \frac{\omega_c}{4\sqrt{2}} = \frac{\pi f_c}{2\sqrt{2}}$$

after some effort. Also note that $\int_{-\infty}^{\infty} h(t)\,dt = H(0) = 1$. Therefore, the noise equivalent bandwidth, from (5.109), is

$$B_N = \frac{\pi f_c}{2\sqrt{2}}\,\text{Hz} = \frac{100\pi}{\sqrt{2}}\,\text{Hz}$$

Problem 5.26

(a) Note that $H_{\max} = 2$, $H(f) = 2$ for $-1 \le f \le 1$, $H(f) = 1$ for $-3 \le f \le -1$ and $1 \le f \le 3$ and is 0 otherwise. Thus

$$\begin{aligned} B_N &= \frac{1}{H_{\max}^2}\int_0^{\infty}|H(f)|^2\,df = \frac{1}{4}\left(\int_0^1 2^2 df + \int_1^2 1^2 df\right) \\ &= \frac{1}{4}(4+1) = 1.25 \text{ Hz} \end{aligned}$$

(b) For this case $H_{\max} = 2$ and the frequency response function is a triangle so that

$$\begin{aligned} B_N &= \frac{1}{H_{\max}^2}\int_0^{\infty}|H(f)|^2\,df = \frac{1}{4}\int_0^{100}\left[2(1-f/100)\right]^2 df \\ &= \int_0^1 (1-v)^2\,dv = -\frac{100}{3}(1-v)^3\Big|_0^1 = 33.33 \text{ Hz} \end{aligned}$$

Problem 5.27

By definition

$$B_N = \frac{1}{H_0^2}\int_0^{\infty}|H(f)|^2\,df = \frac{1}{4}\int_{400}^{600}\left[2\Lambda\left(\frac{f-500}{100}\right)\right]^2 df = 66.67 \text{ Hz}$$

Problem 5.28

a. Use the impulse response method. First, use partial fraction expansion to get the impulse response:

$$H_a(f) = \frac{10}{19}\left(\frac{1}{j2\pi f+1} - \frac{1}{j2\pi f+20}\right) \Leftrightarrow h_a(t) = \frac{10}{19}\left(e^{-t} - e^{-20t}\right)u(t)$$

Thus

$$\begin{aligned}
B_N &= \frac{\int_{-\infty}^{\infty} |h(t)|^2\, dt}{2\left[\int_{-\infty}^{\infty} h(t)\, dt\right]^2} = \frac{\left(\frac{10}{19}\right)^2 \int_0^{\infty} \left(e^{-t} - e^{-20t}\right)^2 dt}{2\left[\left(\frac{10}{19}\right)\int_0^{\infty}\left(e^{-t} - e^{-20t}\right) dt\right]^2} \\
&= \frac{1}{2}\frac{\int_0^{\infty}\left(e^{-2t} - 2e^{-21t} + e^{-40t}\right) dt}{\left[\left(-e^{-t} + \frac{1}{20}e^{-20t}\right)_0^{\infty}\right]^2} \\
&= \frac{1}{2}\frac{\left(-\frac{1}{2}e^{-2t} + \frac{2}{21}e^{-21t} - \frac{1}{40}e^{-40t}\right)_0^{\infty}}{(1 - 1/20)^2} \\
&= \frac{1}{2}\left(\frac{20}{19}\right)^2\left(\frac{1}{2} - \frac{2}{21} + \frac{1}{40}\right) = \frac{1}{2}\left(\frac{20}{19}\right)^2\left(\frac{361}{21 \times 40}\right) \\
&= 0.238 \text{ Hz}
\end{aligned}$$

b. Again use the impulse response method to get $B_N = 0.625$ Hz.

Problem 5.29

(a) Choosing $f_0 = f_1$ moves $-f_1$ right to $f = 0$ and f_1 left to $f = 0$. (b) Choosing $f_0 = f_2$ moves $-f_2$ right to $f = 0$ and f_2 left to $f = 0$. Thus, the baseband spectrum can be written as $S_{\text{LP}}(f) = \frac{1}{2}N_0\Lambda\left(\frac{f}{f_2 - f_1}\right)$. (c) For this case both triangles (left and right) are centered around the origin and they add to give $S_{\text{LP}}(f) = \frac{1}{2}N_0\Pi\left(\frac{f}{f_2 - f_1}\right)$. (d) They are not uncorrelated for any case for an arbitrary delay. However, all cases give quadrature components that are uncorrelated at the same instant.

Problem 5.30

a. By inverse Fourier transformation, the autocorrelation function is

$$R_n(\tau) = \frac{\alpha}{2}e^{-\alpha|\tau|}$$

where $K = \alpha/2$.

b. Use the result from part (a) and the modulation theorem to get

$$R_n^{\prime}(\tau) = \frac{\alpha}{2}e^{-\alpha|\tau|}\cos(2\pi f_0\tau)$$

c. The result is

$$S_{n_c}(f) = S_{n_s}(f) = \frac{\alpha^2}{\alpha^2 + (2\pi f)^2}$$

and

$$S_{n_c n_s}(f) = 0$$

Problem 5.31
(a) The equivalent lowpass power spectral density is given by $S_{n_c}(f) = S_{n_s}(f) = N_0 \Pi\left(\frac{f}{f_2 - f_1}\right)$. The cross-spectral density is $S_{n_c n_s}(f) = 0$. (b) For this case $S_{n_c}(f) = S_{n_s}(f) = \frac{N_0}{2}\Pi\left(\frac{f}{2(f_2-f_1)}\right)$ and the cross-spectral density is

$$S_{n_c n_s}(f) = \begin{cases} -\frac{N_0}{2}, & -(f_2 - f_1) \le f \le 0 \\ \frac{N_0}{2}, & 0 \le f \le (f_2 - f_1) \end{cases}$$

(c) For this case, the lowpass equivalent power spectral densities are the same as for part (b) and the cross-spectral density is the negative of that found in (b). (d) for part (a)

$$R_{n_c n_s}(\tau) = 0$$

For part (b)

$$\begin{aligned}
R_{n_c n_s(\tau)} &= j\left[\int_{-(f_2-f_1)}^{0} -\frac{N_0}{2} e^{j2\pi f\tau} df + \int_0^{f_1-f_2} \frac{N_0}{2} e^{j2\pi f\tau} df\right] \\
&= j\frac{N_0}{2}\left[\left.\frac{-e^{-j2\pi f\tau}}{j2\pi\tau}\right|_{-(f_2-f_1)}^{0} + \left.\frac{e^{-j2\pi f\tau}}{j2\pi\tau}\right|_0^{(f_2-f_1)}\right] \\
&= \frac{N_0}{4\pi\tau}\left(-1 + e^{j2\pi(f_2-f_1)\tau} + e\,-^{j2\pi(f_2-f_1)\tau} - 1\right) \\
&= -\frac{N_0}{4\pi\tau}\left\{2 - 2\cos\left[2\pi\left(f_2 - f_1\right)\tau\right]\right\} \\
&= -\frac{N_0}{\pi\tau}\sin^2\left[\pi\left(f_2 - f_1\right)\tau\right] \\
&= -\left[N_0\pi\left(f_2 - f_1\right)^2\tau\right]\operatorname{sinc}^2\left[\left(f_2 - f_1\right)\tau\right]
\end{aligned}$$

For part (c), the cross-correlation function is the negative of the above.

Problem 5.32
The result is

$$S_{n_2}(f) = \frac{1}{2}S_n(f - f_0) + \frac{1}{2}S_n(f + f_0)$$

so take the given spectrum, translate it to the right by f_0 and to the left by f_0, add the two translated spectra, and divide the result by 2.

Problem 5.33
Define $N_1 = A + N_c$. Then

$$R = \sqrt{N_1^2 + N_s^2}$$

Note that the joint pdf of N_1 and N_s is

$$f_{N_1 N_s}(n_1, n_s) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(n_1 - A)^2 + n_s^2\right]\right\}$$

Thus, the cdf of R is

$$F_R(r) = \Pr(R \le r) = \int\int_{\sqrt{N_1^2+N_s^2} \le r} f_{N_1 N_s}(n_1, n_s)\, dn_1 dn_s$$

Change variables in the integrand from rectangular to polar with

$$n_1 = \alpha \cos\beta \text{ and } n_2 = \alpha \sin\beta$$

Then

$$\begin{aligned} F_R(r) &= \int_0^r \int_0^{2\pi} \frac{\alpha}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(\alpha\cos\beta - A)^2 + (\alpha\sin\beta)^2\right]} d\beta d\alpha \\ &= \int_0^r \frac{\alpha}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(\alpha^2 + A^2\right)} I_0\left(\frac{\alpha A}{\sigma^2}\right) d\alpha \end{aligned}$$

Differentiating with respect to r, we obtain the pdf of R:

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(r^2 + A^2\right)} I_0\left(\frac{rA}{\sigma^2}\right), \quad r \ge 0$$

Problem 5.34
The suggested approach is to apply

$$S_n(f) = \lim_{T\to\infty} \frac{E\left\{\left|\Im\left[x_{2T}(t)\right]\right|^2\right\}}{2T}$$

where $x_{2T}(t)$ is a truncated version of $x(t)$. Thus, let

$$x_{2T}(t) = \sum_{k=-N}^{N} n_k \delta(t - kT_s)$$

The Fourier transform of this truncated waveform is

$$\Im\left[x_{2T}(t)\right] = \sum_{k=-N}^{N} n_k e^{-j2\pi k T_s}$$

Thus,

$$\begin{aligned}
E\left\{\left|\Im\left[x_{2T}(t)\right]\right|^2\right\} &= E\left\{\left|\sum_{k=-N}^{N} n_k e^{-j2\pi k T_s}\right|^2\right\} \\
&= E\left\{\sum_{k=-N}^{N} n_k e^{-j2\pi k T_s} \sum_{l=-N}^{N} n_l e^{j2\pi l T_s}\right\} \\
&= \sum_{k=-N}^{N}\sum_{l=-N}^{N} E\left[n_k n_l\right] e^{-j2\pi(k-l)T_s} \\
&= \sum_{k=-N}^{N}\sum_{l=-N}^{N} R_n(0)\,\delta_{kl} e^{-j2\pi(k-l)T_s} \\
&= \sum_{k=-N}^{N} R_n(0) = (2N+1)\,R_n(0)
\end{aligned}$$

But $2T = 2NT_s$ so that

$$\begin{aligned}
S_n(f) &= \lim_{T\to\infty} \frac{E\left\{\left|\Im\left[x_{2T}(t)\right]\right|^2\right\}}{2T} = \lim_{N\to\infty} \frac{(2N+1)\,R_n(0)}{2NT_s} \\
&= \frac{R_n(0)}{T_s}
\end{aligned}$$

Problem 5.35

To make the process stationary, assume that each sample function is displaced from the origin by a random delay uniform in $(0, T]$. Consider a $2nT$ chunk of the waveform and obtain its Fourier transform as

$$\Im\left[X_{2nT}(t)\right] = \sum_{k=-n}^{n-1} A\tau_0 \operatorname{sinc}(f\tau_0)\, e^{-j2\pi f(\Delta t_k + \tau_0/2 + kT)}$$

Take the magnitude squared and then the expectation to reduce it to

$$E\left[\left|X_{2nT}(f)\right|^2\right] = A^2\tau_0^2 \operatorname{sinc}^2(f\tau_0) \sum_{k=-n}^{n-1}\sum_{m=-n}^{n-1} E\left\{e^{-j2\pi f[\Delta t_k - \Delta t_m + (k-m)T]}\right\}$$

We need the expectation of the inside exponentials which can be separated into the product of the expectations for $k \neq m$, which is

$$E\{\} = \frac{4}{T}\int_0^{T/4} e^{-j2\pi f \Delta t_k} d\Delta t_k = e^{-j2\pi f(k-m)T}\,\mathrm{sinc}^2\left(f\tau/4\right)$$

For $k = m$, the expectation inside the double sum is 1. After considerable manipulation, the psd becomes

$$\begin{aligned} S_X(f) &= \frac{A^2\tau_0^2}{T}\mathrm{sinc}^2\left(f\tau_0\right)\left[1-\mathrm{sinc}\left(fT/4\right)\right] \\ &\quad + \frac{2A^2\tau_0^2\,\mathrm{sinc}^2\left(\tau_0 f\right)}{T}\cos^2\left(\pi f T\right)\left[\lim_{n\to\infty}\frac{\sin^2\left(n\pi f T\right)}{n\sin^2\left(\pi f T\right)}\right] \end{aligned}$$

Examine

$$L = \lim_{n\to\infty}\frac{\sin^2\left(n\pi f T\right)}{n\sin^2\left(\pi f T\right)}$$

It is zero for $f \neq m/T$, m an integer, because the numerator is finite and the denominator goes to infinity. For $f = mT$, the limit can be shown to have the properties of a delta function as $n \to \infty$. Thus

$$S_X(f) = \frac{A^2\tau_0^2}{T}\mathrm{sinc}^2\left(f\tau_0\right)\left[1-\mathrm{sinc}\left(fT/4\right)+2\sum_{m=-\infty}^{\infty}\delta\left(f-m/T\right)\right]$$

Problem 5.36

The result is

$$E\left[y^2(t)\right] = \frac{1}{T^2}\int_t^{t+T}\int_t^{t+T}\left[R_X^2(\tau)+R_X^2(\lambda-\beta)+R_X(\lambda-\beta-\tau)+R_X(\lambda-\beta+\tau)\right]d\lambda d\beta$$

Problem 5.37

1. (a) In the expectation for the cross correlation function, write the derivative as a limit:

$$\begin{aligned}
R_{yy_\mathrm{dot}'}(\tau) &= E\left[y(t)\frac{dy(t+\tau)}{dt}\right] \\
&= E\left\{y(t)\left[\lim_{\epsilon\to 0}\frac{y(t+\tau+\epsilon)-y(t+\tau)}{\epsilon}\right]\right\} \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{E\left[y(t)\,y(t+\tau+\epsilon)\right]-E\left[y(t)\,y(t+\tau)\right]\right\} \\
&= \lim_{\epsilon\to 0}\frac{R_y(\tau+\epsilon)-R_y(\tau)}{\epsilon} \\
&= \frac{dR_y(\tau)}{d\tau}
\end{aligned}$$

(b) The variance of $Y = y(t)$ is $\sigma_Y^2 = N_0 B$. The variance for $Z = \frac{dy}{dt}$ is found from its power spectral density. Using the fact that the transfer function of a differentiator is $j2\pi f$ we get

$$S_Z(f) = (2\pi f)^2 \frac{N_0}{2}\Pi(f/2B)$$

so that

$$\begin{aligned}
\sigma_Z^2 &= 2\pi^2 N_0 \int_{-B}^{B} f^2 df = 2\pi^2 N_0 \frac{f^3}{3}\bigg|_{-B}^{B} \\
&= \frac{4}{3}\pi^2 N_0 B^3
\end{aligned}$$

The two processes are uncorrelated because $R_{yy_\mathrm{dot}'}(0) = \frac{dR_y(\tau)}{d\tau} = 0$ because the derivative of the autocorrelation function of $y(t)$ exists at $\tau = 0$ and therefore must be 0. Thus, the random process and its derivative are independent. Hence, their joint pdf is

$$f_{YZ}(\alpha,\ \beta) = \frac{\exp\left(-\alpha^2/2N_0B\right)}{\sqrt{2\pi N_0 B}}\frac{\exp\left(-\beta^2/2.67\pi^2 N_0 B^3\right)}{\sqrt{2.67\pi^3 N_0 B^3}}$$

(c) No, because the derivative of the autocorrelation function at $\tau = 0$ of such noise does not exist (it is a double-sided exponential).

5.2 Computer Exercises

Computer Exercise 5.1

```
% ce5_1.m: Computes approximations to mean and mean-square ensemble and time
% averages of cosine waves with phase uniformly distributed between 0 and 2*pi
%
clf
f0 = 1;
A = 1;
theta = 2*pi*rand(1,100);
t = 0:.1:1;
X = [];
for n = 1:100
    X = [X; A*cos(2*pi*f0*t+theta(n))];
end
EX = mean(X);
AVEX = mean(X');
EX2 = mean(X.*X);
AVEX2 = mean((X.*X)');
disp(' ')
disp('Sample means (across 100 sample functions)')
disp(EX)
disp(' ')
disp('Typical time-average means (sample functions 1, 20, 40, 60, 80, & 100)')
disp([AVEX(1) AVEX(20) AVEX(40) AVEX(60) AVEX(80) AVEX(100)])
disp(' ')
disp('Sample mean squares')
disp(EX2)
disp(' ')
disp('Typical time-average mean squares')
disp([AVEX2(1) AVEX2(20) AVEX2(40) AVEX2(60) AVEX2(80) AVEX2(100)])
disp(' ')
for n = 1:5
    Y = X(n,:);
    subplot(5,1,n),plot(t,Y),ylabel('X(t,\theta)')
    if n == 1
        title('Plot of first five sample functions')
    end
    if n == 5
```

```
            xlabel('t')
        end
    end
```

A typical run follows:

```
>> ce5_1
Sample means (across 100 sample functions)
Columns 1 through 7
0.0909 0.0292 -0.0437 -0.0998 -0.1179 -0.0909 -0.0292
Columns 8 through 11
0.0437 0.0998 0.1179 0.0909
Typical time-average means (sample functions 1, 20, 40, 60, 80, & 100)
-0.0875 -0.0815 -0.0733 -0.0872 0.0881 0.0273
Sample mean squares
Columns 1 through 7
0.4960 0.5499 0.5348 0.4717 0.4477 0.4960 0.5499
Columns 8 through 11
0.5348 0.4717 0.4477 0.4960
Typical time-average mean squares
0.5387 0.5277 0.5136 0.5381 0.5399 0.4628
```

Computer Exercise 5.2

This is a matter of changing the statement

```
theta = 2*pi*rand(1,100);
```

in the program of Computer Exercise 5.1 to

```
theta = (pi/2)*(rand(1,100) - 0.5);
```

The time-average means will be the same as in Computer Exercise 5.1, but the ensemble-average means will vary depending on the time at which they are computed.

Computer Exercise 5.3

This was done in Computer exercise 4.2 by printing the covariance matrix. The diagonal terms give the variances of the X and Y vectors and the off-diagonal terms give the correlation coefficients. Ideally, they should be zero.

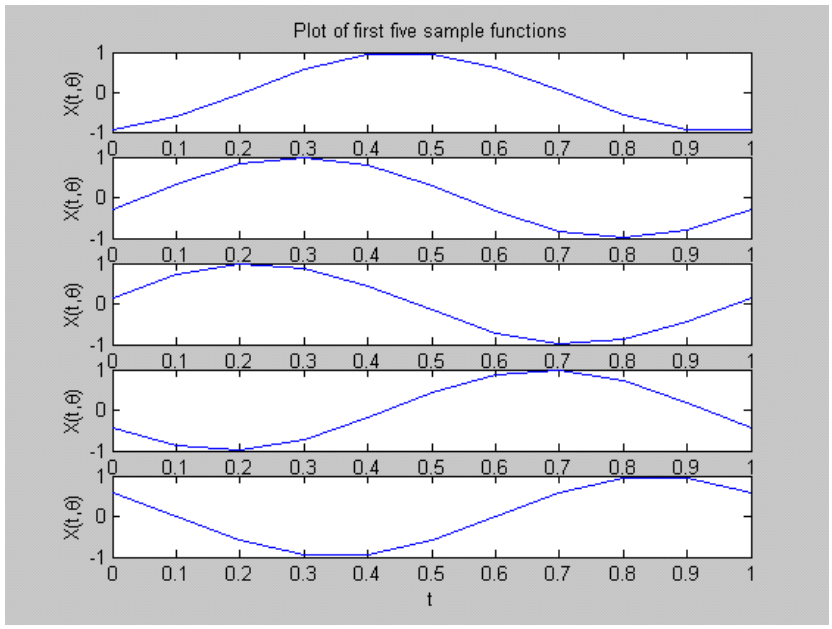

Figure 5.1:

Computer Exercise 5.4

```
% ce5_4.m: plot of Ricean pdf for several values of K
%
clf
A = char('-','–','-.',':','–.','-..');
sigma = input('Enter desired value of sigma ');
r = 0:.1:15*sigma;
n = 1;
KdB = [];
for KK = -10:5:15;
    KdB(n) = KK;
    K = 10.^(KK/10);
    fR = (r/sigma^2).*exp(-(r.^2/(2*sigma^2)+K)).*besseli(0, sqrt(2*K)*(r/sigma));
    plot(r, fR, A(n,:))
    if n == 1
        hold on
        xlabel('r'), ylabel('f_R(r)')
        grid on
```

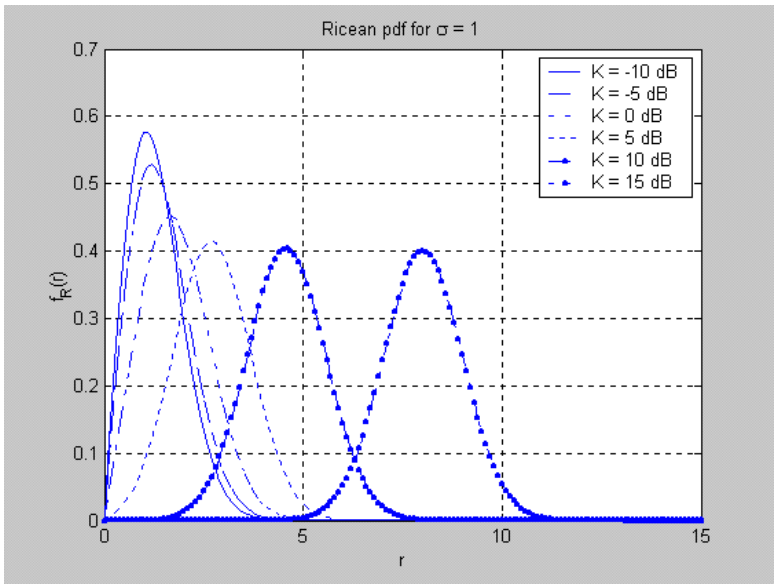

Figure 5.2:

```
    end
    n = n+1;
end
legend(['K = ', num2str(KdB(1)),' dB'],['K = ', num2str(KdB(2)),' dB'],
['K = ', num2str(KdB(3)),' dB'],
    ['K = ', num2str(KdB(4)),' dB'],['K = ', num2str(KdB(5)),' dB'],
['K = ', num2str(KdB(6)),' dB'])
title(['Ricean pdf for \sigma = ', num2str(sigma)])
```

A plot for several values of K is shown below:

Chapter 6

Noise in Modulation Systems

6.1 Problems

Problem 6.1
The signal power at the output of the lowpasss filter is P_T. The noise power is $N_0 B_N$, where B_N is the noise-equivalent bandwidth of the filter. From (5.116), we know that the noise-equivalent bandwidth of an nth order Butterworth filter with 3 dB bandwidth W is

$$B_n(n) = \frac{\pi W/2n}{\sin(\pi/2n)}$$

Thus, the signal-to-noise ratio at the filter output is

$$SNR = \frac{P_T}{N_0 B_N} = \frac{\sin(\pi/2n)}{\pi/2n} \frac{P_T}{N_0 W}$$

so that $f(n)$ is given by

$$f(n) = \frac{\sin(\pi/2n)}{\pi/2n}$$

We can see from the form of $f(n)$ that since $\sin(x) \approx 1$ for $x \ll 1$, $f(\infty) = 1$. Thus for large n, the SNR at the filter output is close to $P_T/N_0 W$. The plot is shown In Figure 6.1.

Problem 6.2
We express $n(t)$ as

$$n(t) = n_c(t)\cos\left[\omega_c t \pm \frac{1}{2}(2\pi W)t + \theta\right] + n_s(t)\sin\left[\omega_c t \pm \frac{1}{2}(2\pi W)t + \theta\right]$$

where we use the plus sign for the USB and the minus sign for the LSB. The received signal is

$$x_r(t) = A_c\left[m(t)\cos(\omega_c t + \theta) \mp \widehat{m}(t)\sin(\omega_c t + \theta)\right]$$

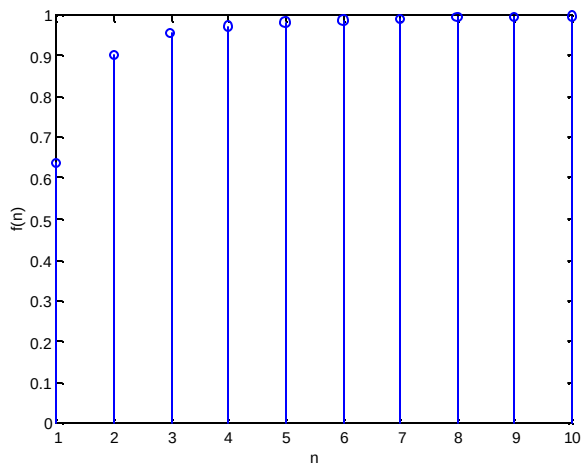

Figure 6.1:

Multiplying $x_r(t) + n(t)$ by $2\cos(\omega_c t + \theta)$ and lowpass filtering yields

$$y_D(t) = A_c m(t) + n_c(t)\cos(\pi W t) \pm n_s(t)\sin(\pi W t)$$

From this expression, we can show that the postdetection signal and noise powers are given by

$$\begin{aligned} S_D &= A_c^2\overline{m^2} \qquad N_D = N_0 W \\ S_T &= A_c\overline{m^2} \qquad N_T = N_0 W \end{aligned}$$

This gives a detection gain of one. The power spectral densities of $n_c(t)$ and $n_s(t)$ are illustrated in Figure 6.2.

Problem 6.3

The received signal and noise are given by

$$x_r(t) = A_c m(t)\cos(\omega_c t + \theta) + n(t)$$

At the output of the predetection filter, the signal and noise powers are given by

$$S_T = \frac{1}{2}A_c^2\overline{m^2} \qquad N_T = \overline{n^2} = N_0 B_T$$

The predetection SNR is

$$(SNR)_T = \frac{A_c^2\overline{m^2}}{2N_0 B_T}$$

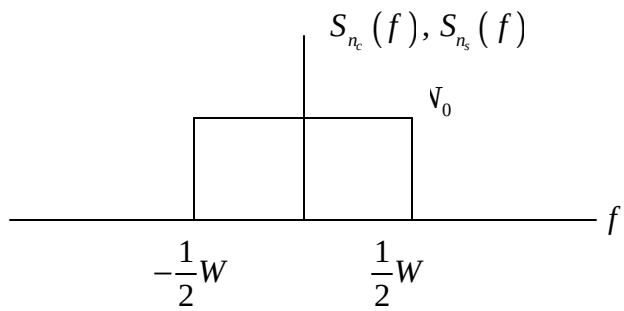

Figure 6.2:

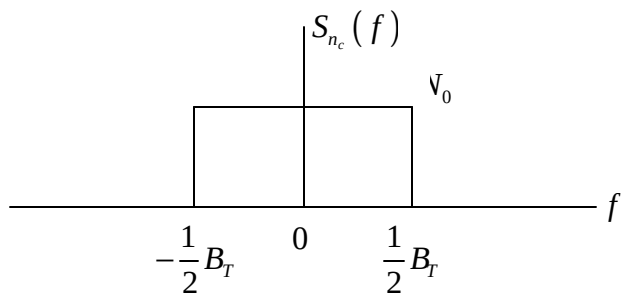

Figure 6.3:

If the postdetection filter passes all of the $n_c(t)$ component, $y_D(t)$ is

$$y_D(t) = A_c m(t) + n_c(t)$$

The output signal power is $A_c^2\overline{m^2}$ and the output noise PSD is shown in Figure 6.3.

Case I: $B_D > \frac{1}{2}B_T$

For this case, all of the noise, $n_c(t)$, is passed by the postdetection filter, and the output noise power is

$$N_D = \int_{-\frac{1}{2}B_D}^{\frac{1}{2}B_T} N_0 df = 2N_0 B_T$$

This yields the detection gain

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2\overline{m^2}/N_0 B_T}{A_c^2\overline{m^2}/2N_0 B_T} = 2$$

Case II: $B_D < \frac{1}{2}B_T$

For this case, the postdetection filter limits the output noise and the output noise power is

$$N_D = \int_{-B_D}^{B_D} N_0 df = 2N_0 B_D$$

This case gives the detection gain

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2\overline{m^2}/2N_0B_D}{A_c^2\overline{m^2}/2N_0B_T} = \frac{B_T}{B_D}$$

Problem 6.4

This problem is identical to Problem 6.3 except that the predetection signal power is

$$S_T = \frac{1}{2}A_c^2\left[1 + a^2\overline{m_n^2}\right]$$

and the postdetection signal power is

$$S_D = A_c^2 a^2\overline{m_n^2}$$

The noise powers do not change.

Case I: $B_D > \frac{1}{2}B_T$

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2\overline{m_n^2}/N_0B_T}{A_c^2\left[1 + a^2\overline{m_n^2}\right]/2N_0B_T} = \frac{2a^2\overline{m_n^2}}{1 + a^2\overline{m_n^2}}$$

Case II: $B_D < \frac{1}{2}B_T$

$$\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2\overline{m_n^2}/2N_0B_D}{A_c^2\left[1 + a^2\overline{m_n^2}\right]/2N_0B_T} = \frac{a^2\overline{m_n^2}}{1 + a^2\overline{m_n^2}} = \frac{B_T}{B_D}$$

Problem 6.5

Since the message signal is sinusoidal

$$m_n(t) = \cos(8\pi t)$$

Thus,

$$\overline{m_n^2} = 0.5$$

The efficiency is therefore given by

$$E_{ff} = \frac{(0.5)(0.8)^2}{1 + (0.5)(0.8)^2} = 0.2424 = 24.24\%$$

From (6.29), the detection gain is

$$\frac{(SNR)_D}{(SNR)_T} = 2E_{ff} = 0.4848 = -3.14\,\text{dB}$$

and the output SNR is, from (6.33),

$$(SNT)_D = 0.2424\frac{P_T}{N_0 W}$$

Relative to baseband, the output SNR is

$$\frac{(SNR)_D}{P_T/N_0 W} = 0.2424 = -6.15\,\text{dB}$$

If the modulation index in increased to 0.9, the efficiency becomes

$$E_{ff} = \frac{(0.5)(0.9)^2}{1 + (0.5)(0.9)^2} = 0.2883 = 28.83\%$$

This gives a detection gain of

$$\frac{(SNR)_D}{(SNR)_T} = 2E_{ff} = 0.5765 = -2.39\,\text{dB}$$

The output SNR is

$$(SNR)_D = 0.2883\frac{P_T}{N_0 W}$$

which, relative to baseband, is

$$\frac{(SNR)_D}{P_T/N_0 W} = 0.2883 = -5.40\,\text{dB}$$

This represents an improvement of 0.75 dB.

Problem 6.6

The first step is to determine the value of M. First we compute

$$P\{X > M\} = \int_M^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy = Q\left(\frac{M}{\sigma}\right) = 0.005$$

This gives

$$\frac{M}{\sigma} = 2.57$$

Thus,

$$M = 2.57\sigma$$

and

$$m_n(t) = \frac{m(t)}{2.57\sigma}$$

Thus, since $\overline{m^2} = \sigma^2$

$$\overline{m_n^2} = \frac{\overline{m^2}}{(2.57\sigma)^2} = \frac{\sigma^2}{6.605\sigma^2} = 0.151$$

Since $a = \frac{1}{2}$, we have

$$E_{ff} = \frac{0.151\left(\frac{1}{2}\right)^2}{1 + 0.151\left(\frac{1}{2}\right)^2} = 3.64\%$$

and the detection gain is

$$\frac{(SNR)_D}{(SNR)_T} = 2E = 0.0728$$

Problem 6.7

The output of the predetection filter is

$$e(t) = A_c[1 + am_n(t)]\cos[\omega_c t + \theta] + r_n(t)\cos[\omega_c t + \theta + \phi_n(t)]$$

The noise function $r_n(t)$ has a Rayleigh *pdf*. Thus

$$f_{R_n}(r_n) = \frac{r}{N} e^{-r^2/2N}$$

where N is the predetection noise power. This gives

$$N = \frac{1}{2}\overline{n_c^2} + \frac{1}{2}\overline{n_s^2} = N_0 B_T$$

From the definition of threshold

$$0.99 = \int_0^{A_c} \frac{r}{N} e^{-r^2/2N} dr$$

which gives

$$0.99 = 1 - e^{-A_c^2/2N}$$

Thus,

$$-\frac{A_c^2}{2N} = 1n\,(0.01)$$

which gives

$$A_c^2 = 9.21\,N$$

The predetection signal power is

$$P_T = \frac{1}{2}A_c\left[1 + a^2\overline{m_n^2}\right] \approx \frac{1}{2}A_c^2\,[1+1] = A_c^2$$

which gives

$$\frac{P_T}{N} = \frac{A_c^2}{N} = 9.21 \approx 9.64\,dB$$

Problem 6.8

Since $m\,(t)$ is a sinusoid, $\overline{m_n^2} = \frac{1}{2}$, and the efficiency is

$$E_{ff} = \frac{\frac{1}{2}a^2}{1 + \frac{1}{2}a^2} = \frac{a^2}{2 + a^2}$$

and the output SNR is

$$(SNR)_D = E_{ff} = \frac{a^2}{2 + a^2}\frac{P_T}{N_0 W}$$

In dB we have

$$(SNR)_{D,dB} = 10\log_{10}\left(\frac{a^2}{2 + a^2}\right) + 10\log_{10}\frac{P_T}{N_0 W}$$

For $a = 0.4$,

$$(SNR)_{D,dB} = 10\log_{10}\frac{P_T}{N_0 W} - 11.3033$$

For $a = 0.5$,

$$(SNR)_{D,dB} = 10\log_{10}\frac{P_T}{N_0 W} - 9.5424$$

For $a = 0.7$,

$$(SNR)_{D,dB} = 10\log_{10}\frac{P_T}{N_0 W} - 7.0600$$

For $a = 0.9$,

$$(SNR)_{D,dB} = 10\log_{10}\frac{P_T}{N_0 W} - 5.4022$$

The plot of $(SNR)_{dB}$ as a function of P_T/N_0W in dB is linear having a slope of one. The modulation index only biases $(SNR)_{D,dB}$ down by an amount given by the last term in the preceding four expressions.

Problem 6.9

Let the predetection filter bandwidth be B_T and let the postdetection filter bandwidth be B_D. The received signal (with noise) at the predetection filter output is represented

$$x_r(t) = A_c[1 + am_n(t)]\cos\omega_c t + n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$$

which can be represented

$$x_r(t) = \{A_c[1 + am_n(t)] + n_c(t)\}\cos\omega_c t - n_s(t)\sin\omega_c t$$

The output of the square-law device is

$$\begin{aligned} y(t) &= \{A_c[1 + am_n(t)] + n_c(t)\}^2\cos^2\omega_c t \\ &\quad -\{A_c[1 + am_n(t)] + n_c(t)\}n_s(t)\cos\omega_c(t) \\ &\quad +n_s^2(t)\sin^2\cos\omega_c(t) \end{aligned}$$

Assuming that the postdetection filter removes the double frequency terms, $y_D(t)$ can be written

$$y_D(t) = \frac{1}{2}\{A_c[1 + am_n(t)] + n_c(t)\}^2 + \frac{1}{2}n_s^2(t)$$

Except for a factor of 2, this is (6.50). We now let $m_n(t) = \cos\omega_m t$ and expand. This gives

$$\begin{aligned} y_D(t) &= \frac{1}{2}A_c^2[1 + a\cos\omega_m t]^2 + A_c n_c(t) \\ &\quad +A_c a n_c(t)\cos\omega_m t + \frac{1}{2}n_c^2(t) + \frac{1}{2}n_s^2(t) \end{aligned}$$

We represent $y_D(t)$ by

$$y_D(t) = z_1(t) + z_2(t) + z_3(t) + z_4(t) + z_5(t)$$

where $z_i(t)$ is the ith term in the expression for $y_D(t)$. We now examine the power spectral density of each term. The first term is

$$z_1(t) = \frac{1}{2}A_c^2\left(1 + \frac{1}{2}a^2\right) + A_c^2 a\cos\omega_m t + \frac{1}{4}A_c^2 a^2\cos 2\omega_m t$$

$$z_2(t) = A_c n_c(t)$$

$$z_3(t) = A_c a n_c(t)\cos\omega_m t$$

$$z_4(t) = \frac{1}{2} n_c^2(t)$$

and

$$z_5(t) = \frac{1}{2} n_s^2(t)$$

A plot of the PSD of each of these terms is illiustrated in Figure 6.4. Adding these five terms together gives the PSD of $y_D(t)$.

Problem 6.10

Assume sinusoidal modulation since sinusoidal modulation was assumed in the development of the square-law detector. For $a = 1$ and $m_n(t) = \cos(2\pi f_m t)$ so that $\overline{m_n^2} = 0.5$, we have, for linear envelope detection (since $P_T/N_0W \gg 1$, we use the coherent result),

$$(SNR)_{D,\ell} = E_{ff}\frac{P_T}{N_0W} = \frac{a^2\overline{m_n^2}}{1 + a^2\overline{m_n^2}}\frac{P_T}{N_0W} = \frac{\frac{1}{2}}{1+\frac{1}{2}}\frac{P_T}{N_0W} = \frac{1}{3}\frac{P_T}{N_0W}$$

For square-law detection we have, from (6.61) with $a = 1$

$$(SNR)_{D,S\ell} = 2\left(\frac{a}{2+a^2}\right)^2\frac{P_T}{N_0W} = \frac{2}{9}\frac{P_T}{N_0W}$$

Taking the ratio

$$\frac{(SNR)_{D,S\ell}}{(SNR)_{D,\ell}} = \frac{\frac{2}{9}\frac{P_T}{N_0W}}{\frac{1}{3}\frac{P_T}{N_0W}} = \frac{2}{3} = -1.76\,\text{dB}$$

This is approximately $-1.8\,\text{dB}$.

Problem 6.11

For the circuit given

$$H(f) = \frac{R}{R + j2\pi fL} = \frac{1}{1 + j\left(\frac{2\pi fL}{R}\right)}$$

and

$$|H(f)|^2 = \frac{1}{1 + \left(\frac{2\pi fL}{R}\right)^2}$$

The output noise power is

$$N = \int_{-\infty}^{\infty}\frac{N_0}{2}\frac{df}{1 + \left(\frac{2\pi fL}{R}\right)^2} = N_0\int_0^{\infty}\frac{1}{1+x^2}\left(\frac{R}{2\pi L}\right)dx = \frac{N_0R}{4L}$$

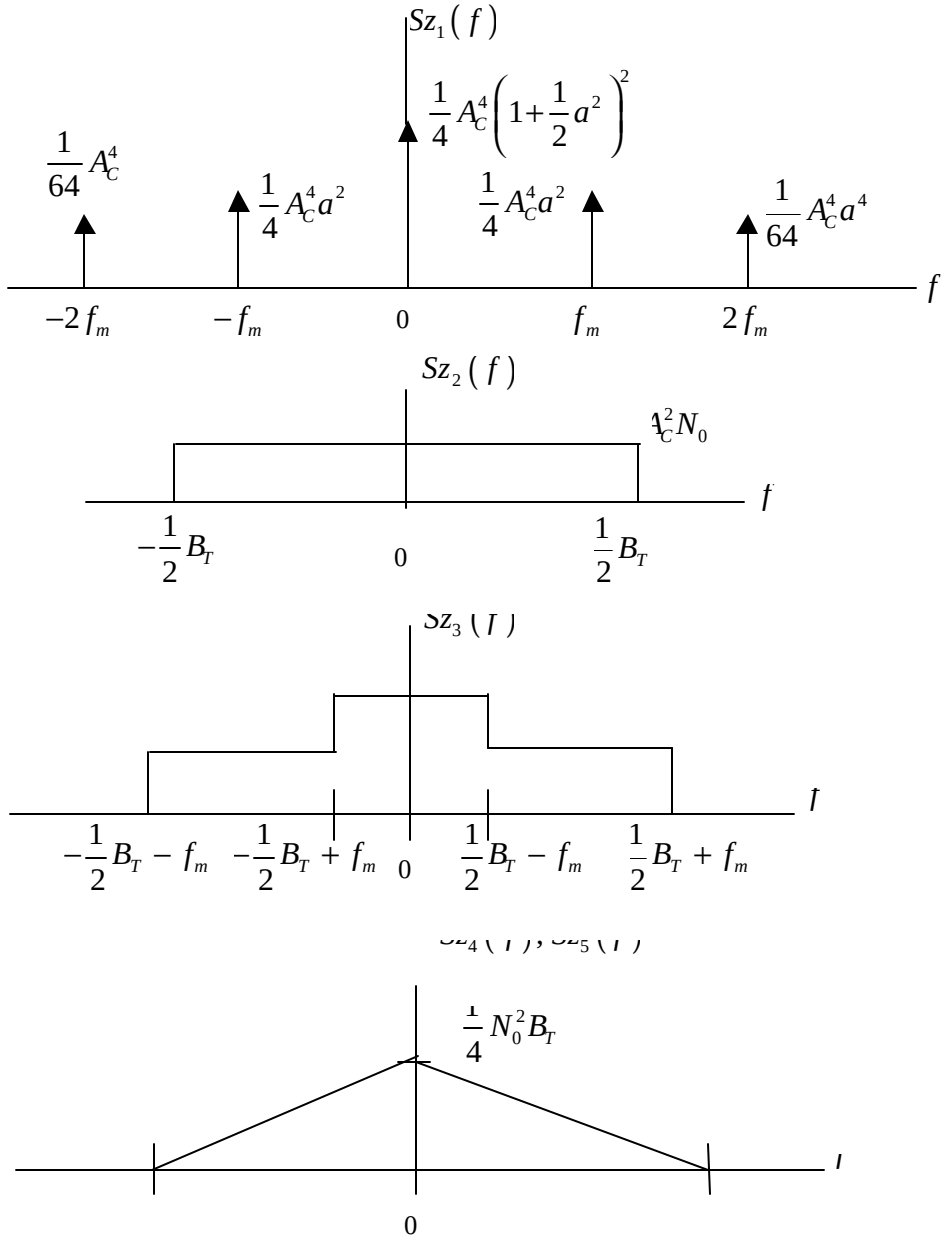
$Sz_1(f)$
$\frac{1}{4}A_C^4\left(1+\frac{1}{2}a^2\right)^2$
$\frac{1}{64}A_C^4$
$\frac{1}{4}A_C^4a^2$
$\frac{1}{4}A_C^4a^2$
$\frac{1}{64}A_C^4a^4$
f
$-2f_m$
$-f_m$
0
f_m
$2f_m$
$Sz_2(f)$
$A_C^2N_0$
f
$-\frac{1}{2}B_T$
0
$\frac{1}{2}B_T$
$Sz_3(f)$
f
$-\frac{1}{2}B_T-f_m$
$-\frac{1}{2}B_T+f_m$
0
$\frac{1}{2}B_T-f_m$
$\frac{1}{2}B_T+f_m$
$\frac{1}{4}N_0^2B_T$
0

Figure 6.4:

The output signal power is

$$S = \frac{1}{2}\frac{A^2R^2}{R^2 + (2\pi f_c L)^2}$$

This gives the signal-to-noise ratio

$$\frac{S}{N} = \frac{2A^2RL}{N_0\left[R^2 + (2\pi f_c L)^2\right]}$$

Problem 6.12
For the RC filter

$$\frac{S}{N} = \frac{2A^2RL}{N_0\left[1 + (2\pi f_c RC)^2\right]}$$

Problem 6.13
The transfer function of the RC highpass filter is

$$H(f) = \frac{R}{R + \frac{1}{j2\pi C}} = \frac{j2\pi fRC}{1 + j2\pi fRC}$$

so that

$$H(f) = \frac{(j2\pi fRC)^2}{1 + (j2\pi fRC)^2}$$

Thus, The PSD at the output of the ideal lowpass filter is

$$S_n(f) = \begin{cases} \frac{N_0}{2}\frac{(2\pi fRC)^2}{1+(2\pi fRC)^2}, & |f| < W \\ 0, & |f| > W \end{cases}$$

The noise power at the output of the ideal lowpass filter is

$$N = \int_{-W}^{W} S_n(f)\,df = N_0\int_0^W \frac{(2\pi fRC)^2}{1 + (2\pi fRC)^2}\,df$$

with $x = 2\pi fRC$, the preceding expression becomes

$$N = \frac{N_0}{2\pi RC}\int_0^{2\pi RCW} \frac{x^2}{1 + x^2}\,dx$$

Since

$$\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2}$$

we can write

$$N = \frac{N_0}{2\pi RC}\left\{\int_0^{2\pi RCW} df - \int_0^{2\pi RCW} \frac{dx}{1+x^2}\right\}$$

or

$$N = \frac{N_0}{2\pi RC}\left(2\pi RCW - \tan^{-1}(2\pi RCW)\right)$$

which is

$$N = N_0 W - \frac{N_0 \tan^{-1}(2\pi RCW)}{2\pi RC}$$

The output signal power is

$$S = \frac{1}{2}A^2 \left|H(f_c)\right|^2 = \frac{A^2}{2}\frac{(2\pi f_c RC)^2}{1+(2\pi f_c RC)^2}$$

Thus, the signal-to-noise ratio is

$$\frac{S}{N} = \frac{A^2}{2N_0}\frac{(2\pi f_c RC)^2}{1+(2\pi f_c RC)^2}\frac{2\pi RC}{2\pi RCW - \tan^{-1}(2\pi RCW)}$$

Note that as $W \to \infty$, $\frac{S}{N} \to 0$.

Problem 6.14

For the case in which the noise in the passband of the postdetection filter is negligible we have

$$\overline{\epsilon_Q^2} = \sigma_\phi^2, \quad \text{SSB and QDSB}$$

and

$$\overline{\epsilon_D^2} = \frac{3}{4}\sigma_\phi^4, \quad \text{DSB}$$

Note that for reasonable values of phase error variance, namely $\sigma_\phi^2 \ll 1$, DSB is much less sensitive to phase errors in the demodulation carrier than SSB or QDSB. Another way of viewing this is to recognize that, for a given acceptable level of $\overline{\epsilon^2}$, the phase error variance for can be much greater for DSB than for SSB or QDSB. The plots follow by simply plotting the two preceding expressions.

Problem 6.15

From the series expansions for $\sin\phi$ and $\cos\phi$ we can write

$$\begin{aligned}\cos\phi &= 1 - \frac{1}{2}\phi^2 + \frac{1}{24}\phi^4 \\ \sin\phi &= \phi - \frac{1}{6}\phi^3\end{aligned}$$

Squaring these, and discarding all terms ϕ^k for $k > 4$, yields

$$\begin{aligned}\cos^2\phi &= 1 - \phi^2 + \frac{1}{3}\phi^4 \\ \sin^2\phi &= \phi^2 - \frac{1}{3}\phi^4\end{aligned}$$

Using (6.70) and recognizing that $m_1(t)$, $m_2(t)$, and $\phi(t)$ are independent, yields

$$\overline{\varepsilon^2} = \sigma_{m_1}^2 - 2\sigma_{m_1}^2\overline{\cos\phi} + \sigma_{m_1}^2\overline{\cos^2\phi} + \sigma_{m_2}^2\overline{\sin^2\phi} + \sigma_n^2$$

Assuming $\phi(t)$ to be a zero-mean process and recalling that

$$\overline{\phi^4} = 3\sigma_\phi^4$$

gives

$$\begin{aligned}\overline{\cos\phi} &= 1 + \frac{1}{2}\sigma_\phi^2 + \frac{1}{8}\sigma_\phi^4 \\ \overline{\cos^2\phi} &= 1 - \sigma_\phi^2 + \sigma_\phi^4 \\ \overline{\sin^2\phi} &= \sigma_\phi^2 - \sigma_\phi^4\end{aligned}$$

This yields

$$\overline{\varepsilon^2} = \sigma_{m_1}^2 - 2\sigma_{m_1}^2\left(1 + \frac{1}{2}\sigma_\phi^2 + \frac{1}{8}\sigma_\phi^4\right) + \sigma_{m_1}^2\left(1 - \sigma_\phi^2 + \sigma_\phi^4\right) + \sigma_{m_2}^2\left(\sigma_\phi^2 - \sigma_\phi^4\right) + \sigma_n^2$$

which can be expressed

$$\overline{\varepsilon^2} = \frac{3}{4}\sigma_{m_1}^2\sigma_\phi^4 + \sigma_{m_2}^2\sigma_\phi^2 - \sigma_{m_2}^2\sigma_\phi^4 + \sigma_n^2$$

For QDSB we let $\sigma_{m_1}^2 = \sigma_{m_2}^2 = \sigma_m^2$. This gives

$$\overline{\varepsilon^2} = \sigma_{m_2}^2\left(\sigma_\phi^2 - \frac{1}{4}\sigma_\phi^4\right) + \sigma_n^2$$

For $\sigma_\phi^2 >> \sigma_\phi^4$, we have

$$\overline{\varepsilon^2} = \sigma_m^2\sigma_\phi^2 + \sigma_n^2$$

which yields (6.73). For DSB, we let $\sigma_{m_1}^2 = \sigma_m^2$ and $\sigma_{m_2}^2 = 0$. This gives (6.79)

$$\overline{\varepsilon^2} = \frac{3}{4}\sigma_m^2\sigma_\phi^4 + \sigma_n^2$$

Problem 6.16
From (6.73) and (6.79), we have

$$\begin{aligned} 0.05 &= \sigma_\phi^2 + \frac{\sigma_n^2}{\sigma_m^2}, \quad \text{SSB} \\ 0.05 &= \frac{3}{4}\left(\sigma_\phi^2\right)^2 + \frac{\sigma_n^2}{\sigma_m^2}, \quad \text{DSB} \end{aligned}$$

Thus we have the following signal-to-noise ratios

$$\begin{aligned} \frac{\sigma_m^2}{\sigma_n^2} &= \frac{1}{0.05 - \sigma_\phi^2}, \quad \text{SSB} \\ \frac{\sigma_m^2}{\sigma_n^2} &= \frac{1}{0.05 - \frac{3}{4}\left(\sigma_\phi^2\right)^2}, \quad \text{DSB} \end{aligned}$$

The SSB characteristic has an asymptote defined by

$$\sigma_\phi^2 = 0.05$$

and the DSB result has an asymptote defined by

$$\frac{3}{4}\left(\sigma_\phi^2\right)^2 = 0.05$$

or

$$\sigma_\phi^2 = 0.258$$

The curves are shown in Figure 6.5. The appropriate operating regions are above and to the left of the curves. It is clear that DSB has the larger operating region.

Problem 6.17
Since we have a DSB system

$$\overline{\varepsilon_N^2} = \frac{3}{4}\sigma_\phi^4 + \frac{\sigma_n^2}{\sigma_m^2}$$

Let the bandwidth of the predetection filter be B_T and let the bandwidth of the pilot filter be

$$B_p = \frac{B_T}{\alpha}$$

This gives

$$\frac{\sigma_n^2}{\sigma_m^2} = \frac{N_0 B_T}{\sigma_m^2} = \frac{B_T}{B_p}\frac{N_0 B_p}{\sigma_m^2} = \frac{\alpha}{\rho}$$

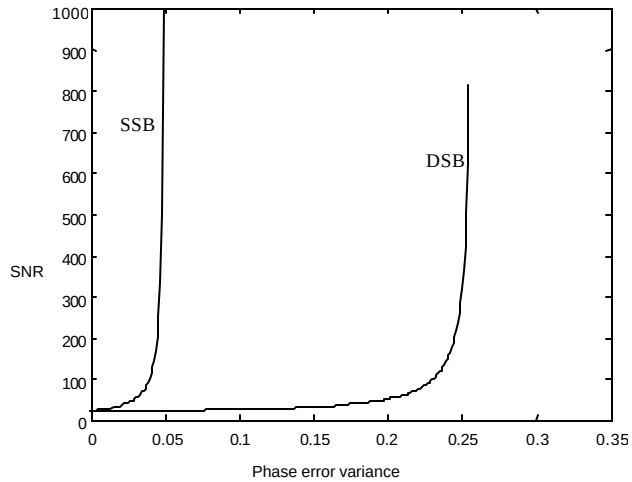

Figure 6.5:

From (6.85) and (6.86), we have

$$\sigma_\phi^2 = \frac{1}{k^2}\frac{1}{2\rho}$$

so that

$$\overline{\varepsilon_N^2} = \frac{3}{16}\frac{1}{k^4\rho^2} + \frac{\alpha}{\rho}$$

For an SNR of 15 dB

$$\rho = 10^{1.5} = 31.623$$

Using this value for ρ and with $k = 4$, we have

$$\overline{\varepsilon_N^2} = 7.32\left(10^{-7}\right) + 0.032\alpha$$

The plot is obviously linear in α with a slope of 0.032. The bias, 7.32×10^{-7}, is negligible. Note that for $k > 1$ and reasonable values of the pilot signal-to-noise ratio, ρ, the first term (the bias), which arises from pilot phase jitter, is negligible. The performance is limited by the additive noise.

Problem 6.18

The mean-square error

$$\overline{\varepsilon^2(A,\tau)} = E\left\{[y(t) - Ax(t-\tau)]^2\right\}$$

can be written

$$\overline{\varepsilon^2(A,\tau)} = E\left\{y^2(t) - 2Ax(t-\tau)y(t) + A^2x^2(t-\tau)\right\}$$

In terms of autocorrelation and cross correlation functions, the mean-square error is

$$\overline{\varepsilon^2(A,\tau)} = R_y(0) - 2AR_{xy}(\tau) + A^2R_x(0)$$

Letting $P_y = R_y(0)$ and $P_x = R_x(0)$ gives

$$\overline{\varepsilon^2(A,\tau)} = P_y - 2AR_{xy}(\tau) + A^2P_x$$

In order to minimize $\overline{\varepsilon^2}$ we choose $\tau = \tau_m$ such that $R_{xy}(\tau)$ is maximized. This follows since the crosscorrelation term is negative in the expression for $\overline{\varepsilon^2}$. Therefore,

$$\overline{\varepsilon^2(A,\tau_m)} = P_y - 2AR_{xy}(\tau_m) + A^2P_x$$

The gain A_m is determined from

$$\overline{\frac{d\varepsilon^2}{dA}} = -2AR_{xy}(\tau_m) + A^2P_x = 0$$

which yields

$$A_m = \frac{R_{xy}(\tau_m)}{P_x}$$

This gives the mean-square error

$$\overline{\varepsilon^2(A_m,\tau_m)} = P_y - 2\frac{R_{xy}^2(\tau_m)}{P_x} + \frac{R_{xy}^2(\tau_m)}{P_x}$$

which is

$$\overline{\varepsilon^2(A_m,\tau_m)} = P_y - \frac{P_{xy}^2(\tau_m)}{P_x}$$

The output signal power is

$$S_D = E\left\{[A_mx(t-\tau_m)]^2\right\} = A_m^2P_x$$

which is

$$S_D = \frac{R_{xy}^2(\tau_m)}{P_x} = \frac{R_{xy}^2(\tau_m)}{R_x(0)}$$

Since N_D is the error $\overline{\varepsilon^2(A_m,\tau_m)}$ we have

$$\frac{S_D}{N_D} = \frac{R_{xy}^2(\tau_m)}{R_x(0)\,R_y(0) - R_{xy}^2(\tau_m)}$$

Note: The gain and delay of a linear system is often defined as the magnitude of the transfer and the slope of the phase characteristic, respectively. The definition of gain and delay suggested in this problem is useful when the magnitude response is not linear over the frequency range of interest.

Problem 6.19

The single-sided spectrum of a stereophonic FM signal and the noise spectrum is shown in Figure 6.6. The two-sided noise spectrum is given by

$$S_{nF}(f) = \frac{K_D^2}{A_C^2} N_0 f, \quad -\infty < f < \infty$$

The predetection noise powers are easily computed. For the $L+R$ channel

$$P_{n,L+R} = 2\int_0^{15,000} \frac{K_D^2}{A_C^2} N_0 \ f^2 df = 2.25\left(10^{12}\right)\frac{K_D^2}{A_C^2} N_0$$

For the $L-R$ channel

$$P_{n,L-R} = 2\int_{23,000}^{53,000} \frac{K_D^2}{A_C^2} N_0 \ f^2 df = 91.14\left(10^{12}\right)\frac{K_D^2}{A_C^2} N_0$$

Thus, the noise power on the $L-R$ channel is over 40 times the noise power in the $L+R$ channel. After demodulation, the difference will be a factor of 20 because of 3 dB detection gain of coherent demodulation. Thus, the main source of noise in a stereophonic system is the $L-R$ channel. Therefore, in high noise environments, monophonic broadcasting is preferred over stereophonic broadcasting.

Problem 6.20

The received FDM spectrum is shown in Figure 6.7. The kth channel signal is given by

$$x_k(t) = A_k m_k(t) \cos 2\pi k f_1 t$$

Since the guardbands have spectral width $4W$, $fk = 6kW$ and the kth channel occupies the frequency band

$$(6k-1)W \leq f \leq (6k+1)W$$

Since the noise spectrum is given by

$$S_{nF} = \frac{K_D^2}{A_C^2} N_0 \ f^2, \quad |f| < B_T$$

The noise power in the kth channel is

$$N_k = B\int_{(6k-1)W}^{(6k+1)W} f^2 df = \frac{BW^3}{3}\left[(6k+1)^3 - (6k-1)^3\right]$$

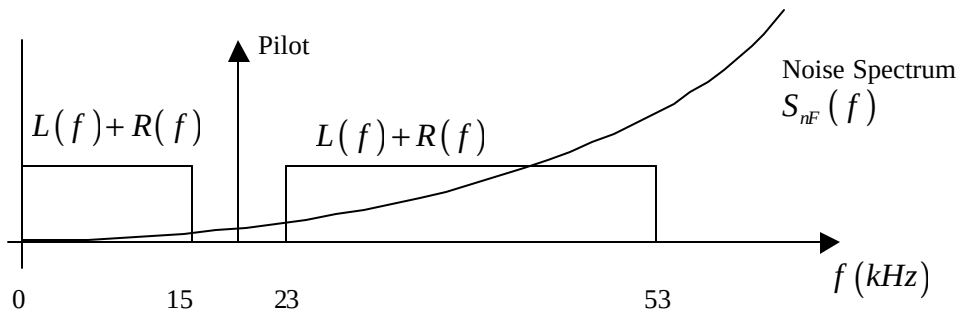

Figure 6.6:

where B is a constant. This gives

$$N_k = \frac{BW^3}{3}\left(216k^2 + 2\right) \cong 72BW^3k^2$$

The signal power is proportional to A_k^2. Thus, the signal-to-noise ratio can be expressed as

$$(SNR)_D = \frac{\lambda A_k^2}{k^2} = \lambda\left(\frac{A_k}{k}\right)^2$$

where λ is a constant of proportionality and is a function of K_D,W,A_C,N_0. If all channels are to have equal $(SNR)_D$, A_k/k must be a constant, which means that A_k is a linear function of k. Thus, if A_1 is known, $A_k = k\ A_1, k = 1, 2, ..., 7$. A_1 is fixed by setting the SNR of Channel 1 equal to the SNR of the unmodulated channel. The noise power of the unmodulated channel is

$$N_0 = B\int_0^W f^2 df = \frac{B}{3}W^3$$

yielding the signal-to-noise ratio

$$(SNR)_D = \frac{P_0}{\frac{B}{3}W^3}$$

where P_0 is the signal power.

Problem 6.21

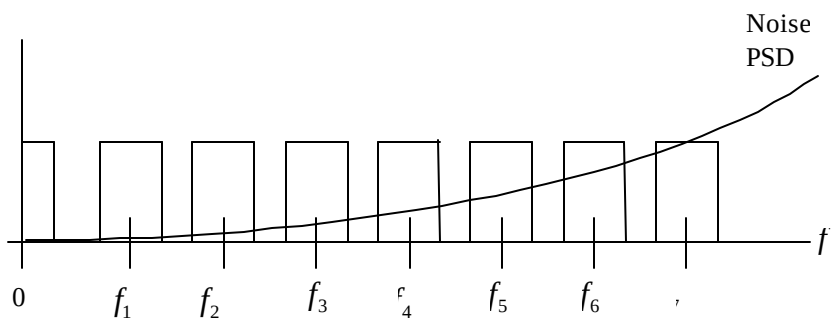

Figure 6.7:

From (6.132) and (6.119), the required ratio is

$$R = \frac{2\frac{K_D^2}{A_C^2} N_0 f_3^3 \left(\frac{W}{f_3} - \tan^{-1}\frac{W}{f_3}\right)}{\frac{2}{3}\frac{K_D^2}{A_C^2} N_0 W^3}$$

or

$$R = 3\left(\frac{f_3}{W}\right)^3 \left(\frac{W}{f_3} - \tan^{-1}\frac{W}{f_3}\right)$$

This is shown in Figure 6.8.

For $f_3 = 2.1kHz$ and $W = 15kHz$, the value of R is

$$R = 3\left(\frac{2.1}{15}\right)^2 \left(\frac{15}{2.1} - \tan^{-1}\frac{15}{2.1}\right) = 0.047$$

Expressed in dB this is

$$R = 10\log_{10}(0.047) = -13.3 \quad \text{dB}$$

The improvement resulting from the use of preemphasis and deemphasis is therefore 13.3 dB. Neglecting the $\tan^{-1}(W/f_3)$ gives an improvement of $21 - 8.75 = 12.25$ dB. The difference is approximately 1 dB.

Problem 6.22

From the plot of the signal spectrum it is clear that

$$k = \frac{A}{W^2}$$

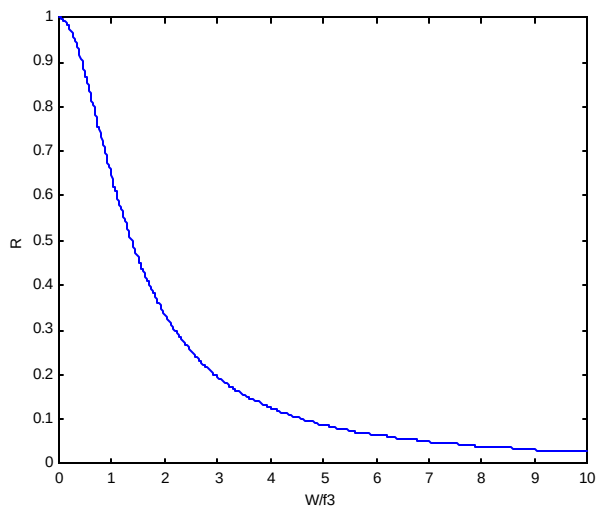

Figure 6.8:

Thus the signal power is

$$S = 2\int_0^W \frac{A}{W^2} f^2 df = \frac{2}{3}AW$$

The noise power is $N_0 B$. This yields the signal-to-noise ratio

$$(SNR)_1 = \frac{2}{3}\frac{AW}{N_0 B}$$

If B is reduced to W, the SNR becomes

$$(SNR)_2 = \frac{2}{3}\frac{A}{N_0}$$

This increases the signal-to-noise ratio by a factor of B/W.

Problem 6.23

From the definition of the signal we have

$$\begin{aligned} x(t) &= A\cos 2\pi f_c t \\ \frac{dx}{dt} &= -2\pi f_c A \sin 2\pi f_c t \\ \frac{d^2x}{dt^2} &= -(2\pi f_c)^2 A\cos 2\pi f_c t \end{aligned}$$

The signal component of $y(t)$ therefore has power

$$S_D = \frac{1}{2}A^2 (2\pi f_c)^4 = 8A^2\pi^4 f_c^4$$

The noise power spectral density at $y(t)$ is

$$S_n(f) = \frac{N_0}{2}(2\pi f)^4$$

so that the noise power is

$$N_D = \frac{N_0}{2}(2\pi)^4 \int_{-W}^{W} f^4 df = \frac{16}{5} N_0 \pi^4 W^5$$

Thus, the signal-to-noise ratio at $y(t)$ is

$$(SNR)_D = \frac{5}{2}\frac{A^2}{N_0 W}\left(\frac{f_c}{W}\right)^4$$

Problem 6.24

Since the signal is integrated twice and then differentiated twice, the output signal is equal to the input signal. The output signal is

$$y_s(t) = A\cos 2\pi f_c t$$

and the output signal power is

$$S_D = \frac{1}{2}A^2$$

The noise power is the same as in the previous problem, thus

$$N_D = \frac{16}{5} N_0 \pi^4 W^5$$

This gives the signal-to-noise ratio

$$(SNR)_D = \frac{5}{32\pi^4}\frac{A^2}{N_0 W^5}$$

Problem 6.25

The signal-to-noise ratio at $y(t)$ is

$$(SNR)_D = \frac{2A}{N_0}\left(\frac{f_3}{W}\right)\tan^{-1}\frac{W}{f_3}$$

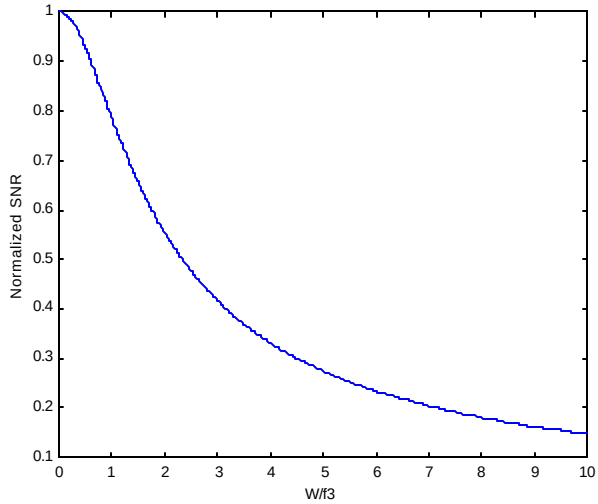

Figure 6.9:

The normalized $(SNR)_D$, defined by $(SNR)_D/(2A?N_0)$, is illustrated in Figure 6.9.

Problem 6.26

The instantaneous frequency deviation in Hz is

$$\delta f = f_d m(t) = x(t)$$

where $x(t)$ is a zero-mean Gaussian process with variance $\sigma_x^2 = f_d^2 \sigma_m^2$. Thus,

$$\overline{|\delta f|} = \overline{|x|} = \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2} dx$$

Recognizing that the integrand is even gives

$$\overline{|\delta f|} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_x} \int_0^{\infty} x e^{-x^2/2\sigma_x^2} dx = \sqrt{\frac{2}{\pi}} \sigma_x$$

Therefore,

$$\overline{|\delta f|} = \sqrt{\frac{2}{\pi}} f_d \sigma_m$$

Substitution into (6.148) gives

$$(SNR)_D = \frac{3\left(\frac{f_d}{W}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W}}{1 + 2\sqrt{3}\frac{B_T}{W} Q\left[\sqrt{\frac{A_C^2}{N_0 B_T}}\right] + 6\sqrt{\frac{2}{\pi}}\frac{f_d \sigma_m}{W} \exp\left[\frac{-A_C^2}{2N_0 B_T}\right]}$$

The preceding expression can be placed in terms of the deviation ratio, D, by letting

$$D = \frac{f_d}{W}$$

and

$$B_T = 2\,(D+1)\,W$$

Problem 6.27

(Note: This problem was changed after the first printing of the book. The new problem, along with the solution follow.)

Assume a PCM system in which the the bit error probability P_b is sufficiently small to justify the approximation that the word error pobability is $P_w \approx nP_b$. Also assume that the threshold value of the signal-to-noise ratio, defined by (6.175), occurs when the two denominator terms are equal, i.e., the effect of quantizating errors and word errors are equivalent. Using this assumption derive the threshold value of P_T/N_0B_p in dB for $n = 4, 8,$ and 12. Compare the results with Figure 6.22 derived in Example 6.5.

With $P_w = nP_b$, equating the two terms in the denominator of (6.175) gives

$$2^{-2n} = nP_b(1 - 2^{-2n})$$

Solving for P_b we have

$$P_n = \frac{1}{n}\frac{2^{-2n}}{1 - 2^{-2n}} \approx \frac{1}{n}2^{-2n}$$

From (6.178)

$$\exp\left[-\frac{P_T}{2N_0B_p}\right] = 2P_b \approx \frac{1}{n}2^{(1-2n)}$$

Solving for P_T/N_0B_p gives, at threshold,

$$\frac{P_T}{N_0B_p} \approx -2\ln\left[\frac{1}{n}2^{(1-2n)}\right] = 2(2n-1)\ln(2) + 2\ln(2)$$

The values of P_T/N_0B_p for $n = 4, 8,$ and 12 are given in Table 6.1. Comparison with Figure 6.22 shows close agreement.

Table 6.1:

n	**Threshold value of** P_T/N_0B_p
4	10.96 dB
8	13.97 dB
12	15.66 dB

Problem 6.28

The peak value of the signal is 7.5 so that the signal power is

$$S_D = \frac{1}{2}(7.5)^2 = 28.125$$

If the A/D converter has a wordlength n, there are 2^n quantizing levels. Since these span the peak-to-peak signal range, the width of each quantization level is

$$S_D = \frac{15}{2^n} = 15\left(2^{-n}\right)$$

This gives

$$\overline{\varepsilon^2} = \frac{1}{12}S_D^2 = \frac{1}{12}(15)^2\left(2^{-2n}\right) = 18.75\left(2^{-2n}\right)$$

This results in the signal-to-noise ratio

$$SNR = \frac{S_D}{\overline{\varepsilon^2}} = \frac{28.125}{18.75}\left(2^{2n}\right) = 1.5\left(2^{2n}\right)$$

Chapter 7

Binary Data Communication

7.1 Problem Solutions

Problem 7.1
The signal-to-noise ratio is

$$z = \frac{A^2 T}{N_0} = \frac{A^2}{N_0 R}$$

Trial and error using the asymptotic expression $Q(x) \approx \exp\left(-x^2/2\right) / \left(\sqrt{2\pi}x\right)$ shows that

$$P_E = Q\left(\sqrt{2z}\right) = 10^{-5} \text{ for } z = 9.58 \text{ dB} = 9.078$$

Thus

$$\frac{A^2}{N_0 R} = 9.078$$

or

$$\begin{aligned} A &= \sqrt{9.078 N_0 R} \\ &= \sqrt{9.078 \times 10^{-7} \times 20000} \\ &= 0.135 \text{ V} \end{aligned}$$

Problem 7.2
The bandwidth should be equal to the data rate to pass the main lobe of the signal spectrum. By trial and error, solving the following equation for z,

$$P_E|_{\text{desired}} = Q\left(\sqrt{2\, z|_{\text{required}}}\right),\ z = E_b/N_0$$

we find that $z|_{\text{required}} = 6.78$ dB (4.76 ratio), 8.39 dB (6.90 ratio), 9.58 dB (9.08 ratio), 10.53 dB (11.3 ratio) for $P_E|_{\text{desired}} = 10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6}, respectively. The required signal power is

$$P_{s,\text{ required}} = E_{b,\text{ required}}/T_{\text{symbol}} = E_{b,\text{ required}}R = z|_{\text{required}} N_0 R$$

For example, for $P_E = 10^{-3}$ and $R = 1000$ bits/second, we get

$$P_{s,\text{ required}} = z|_{\text{required}} N_0 R = 4.76 \times 10^{-6} \times 1000 = 4.76 \times 10^{-3} \text{ W}$$

Similar calculations allow the rest of the table to be filled in, giving the following results.

R, bps	$P_E = 10^{-3}$	$P_E = 10^{-4}$	$P_E = 10^{-5}$	$P_E = 10^{-6}$
$1,000$	4.76×10^{-3} W	6.9×10^{-3} W	9.08×10^{-3} W	11.3×10^{-3} W
$10,000$	4.76×10^{-2} W	6.9×10^{-2} W	9.08×10^{-2} W	11.3×10^{-2} W
$100,000$	4.76×10^{-1} W	6.9×10^{-1} W	9.08×10^{-1} W	1.13 W

Problem 7.3
(a) For $R = B = 5,000$ bps, we have

$$\begin{aligned} \text{signal power} &= A^2 = z|_{\text{required}} N_0 R \\ &= 10^{10.53/10} \times 10^{-6} \times 5,000 \\ &= 11.3 \times 5 \times 10^{-3} = 0.0565 \text{ W} \end{aligned}$$

where $z|_{\text{required}} = 10.53$ dB from Problem 7.2.
(b) For $R = B = 50,000$ bps, the result is

$$\text{signal power} = A^2 = 0.565 \text{ W}$$

(c) For $R = B = 500,000$ bps, we get

$$\text{signal power} = A^2 = 5.65 \text{ W}$$

(d) For $R = B = 1,000,000$ bps, the result is

$$\text{signal power} = A^2 = 11.3 \text{ W}$$

Problem 7.4
The decision criterion is

$$\begin{aligned} V &> \varepsilon, \text{ choose } +A \\ V &< \varepsilon, \text{ choose } -A \end{aligned}$$

where

$$V = \pm AT + N$$

N is a Gaussian random variable of mean zero and variance $\sigma^2 = N_0T/2$. Because $P(A \text{ sent}) = P(-A \text{ sent}) = 1/2$, it follows that

$$P_E = \frac{1}{2}P(-AT + N > \varepsilon \,|\, -A \text{ sent}) + \frac{1}{2}P(AT + N < \varepsilon \,|\, A \text{ sent}) = \frac{1}{2}(P_1 + P_2)$$

where

$$\begin{aligned} P_1 &= \int_{AT+\varepsilon}^{\infty} \frac{e^{-\eta^2/N_0T}}{\sqrt{\pi N_0 T}} d\eta = Q\left[\sqrt{\frac{2A^2T}{N_0}} + \sqrt{\frac{2\varepsilon^2}{N_0T}}\right] \\ &= Q\left(\sqrt{2z} + \sqrt{2}\epsilon/\sigma\right),\ \sigma = N_0T,\ z = \frac{A^2T}{N_0} \end{aligned}$$

Similarly

$$\begin{aligned} P_2 &= \int_{-\infty}^{-AT+\varepsilon} \frac{e^{-\eta^2/N_0T}}{\sqrt{\pi N_0 T}} d\eta = \int_{AT-\varepsilon}^{\infty} \frac{e^{-\eta^2/N_0T}}{\sqrt{\pi N_0 T}} d\eta \\ &= Q\left[\sqrt{\frac{2A^2T}{N_0}} - \sqrt{\frac{2\varepsilon^2}{N_0T}}\right] = Q\left(\sqrt{2z} - \sqrt{2}\epsilon/\sigma\right) \end{aligned}$$

Use the approximation

$$Q(u) = \frac{e^{-u^2/2}}{u\sqrt{2\pi}}$$

to evaluate

$$\frac{1}{2}(P_1 + P_2) = 10^{-6}$$

by solving iteratively for z for various values of ε/σ. The results are given in the table below.

ε/σ	z_{dB} for $P_E = 10^{-6}$	ε/σ	z_{dB} for $P_E = 10^{-6}$
0	10.54	0.6	11.34
0.2	10.69	0.8	11.67
0.4	11.01	1.0	11.98

Problem 7.5

Use the $z|_{\text{required}}$ values found in the solution of Problem 7.2 along with $z|_{\text{required}} = A^2/N_0R$ to get $R = A^2/\, z|_{\text{required}}\, N_0$. Substitute $N_0 = 10^{-6}$ V^2/Hz, $A = 20$ mV to get $R = 400/\, z|_{\text{required}}$. This results in the values given in the table below.

P_E	$z\|_{\text{required}}$, dB (ratio)	R, bps
10^{-4}	8.39 (6.9)	57.97
10^{-5}	9.58 (9.08)	43.05
10^{-6}	10.53 (11.3)	35.4

Problem 7.6

The integrate-and-dump detector integrates over the interval $[0,\ T-\delta]$ so that the SNR is

$$z' = \frac{A^2\,(T-\delta)}{N_0}$$

instead of $z = A^2T/N_0$. Thus

$$\frac{z'}{z} = 1 - \frac{T}{\delta}$$

and the degradation in z in dB is

$$D = -10\log_{10}\left(1 - \frac{T}{\delta}\right)$$

Using $\delta = 10^{-6}$ s and the data rates given, we obtain the following values of D: (a) for $1/T = R = 10$ kbps, $D = 0.04$ dB; (b) for $R = 50$ kbps, $D = 0.22$ dB; (c) for $R = 100$ kbps, $D = 0.46$ dB.

Problem 7.7

(a) For the pulse sequences $(-A,\ -A)$ and $(A,\ A)$, which occur half the time, no degradation results from timing error, so the error probability is $Q\left(\sqrt{2E_b/N_0}\right)$. The error probability for the sequences $(-A,\ A)$ and $(A,\ -A)$, which occur the other half the time, result in the error probability $Q\left[\sqrt{2E_b/N_0}\,(1 - 2\,|\Delta T|\,/T)\right]$ (sketches of the pulse sequences with the integration interval offset from the transition between pulses is helpful here). Thus, the average error probability is the given result in the problem statement. Plots are shown in Figure 7.1 for the giving timing errors.

(b) The plots given in Figure 7.1 for timing errors of 0 and 0.15 indicate that the degradation at $P_E = 10^{-6}$ is about 2.8 dB.

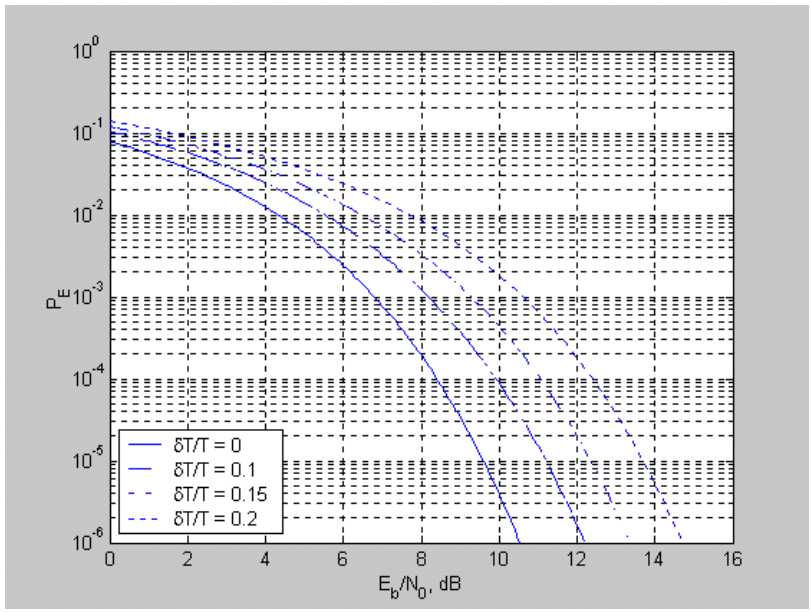

Figure 7.1:

Problem 7.8

(a) The impulse response of the filter is

$$h(t) = (2\pi f_3)\, e^{-2\pi f_3 t} u(t)$$

and the step response is the integral of the impulse response, or

$$a(t) = \left(1 - e^{-2\pi f_3 t}\right) u(t)$$

Thus, for a step of amplitude A, the signal out at $t = T$ is

$$s_0(T) = A\left(1 - e^{-2\pi f_3 T}\right)$$

The power spectral density of the noise at the filter output is

$$S_n(f) = \frac{N_0/2}{1 + (f/f_3)^2}$$

so that the variance of the noise component at the filter output is

$$E\left[N^2\right] = \int_{-\infty}^{\infty} S_n(f)\, df = \frac{N_0 \pi f_3}{2} = N_0 B_N$$

where B_N is the noise equivalent bandwidth of the filter. Thus

$$\text{SNR} = \frac{s_0^2(T)}{E[N^2]} = \frac{2A^2\left(1 - e^{-2\pi f_3 T}\right)^2}{N_0 \pi f_3}$$

(b) To maximize the SNR, differentiate with respect to f_3 and set the result equal to 0. Solve the equation for f_3. This results in the equation

$$4\pi f_3 T e^{-2\pi f_3 T} - 1 + e^{-2\pi f_3 T} = 0$$

Let $\alpha = 2\pi f_3 T$ to get the equation

$$(2\alpha + 1)\, e^{-\alpha} = 1$$

A numerical solution results in $\alpha \approx 1.25$. Thus

$$f_{3,\text{ opt}} = \frac{1.25}{2\pi T} = 0.199/T = 0.199R$$

Problem 7.9

As in the derivation in the text for antipodal signaling,

$$\text{var}[N] = \frac{1}{2} N_0 T$$

The output of the integrator at the sampling time is

$$V = \begin{cases} AT + N, & A \text{ sent} \\ N, & 0 \text{ sent} \end{cases}$$

The probabilities of error, given these two signaling events, are

$$\begin{aligned} P(\text{error} \mid A \text{ sent}) &= P\left(AT + N < \frac{1}{2}AT\right) \\ &= P\left(N < -\frac{1}{2}AT\right) \\ &= \int_{-\infty}^{-AT/2} \frac{e^{-\eta^2/N_0 T}}{\sqrt{\pi N_0 T}} d\eta \\ &= Q\left[\sqrt{\frac{A^2 T}{2N_0}}\right] \end{aligned}$$

Similarly,

$$\begin{aligned} P\left(\text{error} \mid 0 \text{ sent}\right) &= P\left(N > \frac{1}{2}AT\right) \\ &= \int_{AT/2}^{\infty} \frac{e^{-\eta^2/N_0T}}{\sqrt{\pi N_0 T}} d\eta \\ &= Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

Thus

$$\begin{aligned} P_E &= \frac{1}{2}P\left(\text{error} \mid A \text{ sent}\right) + \frac{1}{2}P\left(\text{error} \mid 0 \text{ sent}\right) \\ &= Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

But, the average signal energy is

$$\begin{aligned} E_{\text{ave}} &= P\left(A \text{ sent}\right) \times A^2T + P\left(0 \text{ sent}\right) \times 0 \\ &= \frac{1}{2}A^2T \end{aligned}$$

Therefore, the average probability of error is

$$P_E = Q\left[\sqrt{\frac{E_{\text{ave}}}{N_0}}\right]$$

Problem 7.10

Substitute (7.27) and (7.28) into

$$\begin{aligned} P_E &= pP\left(E \mid s_1\left(t\right)\right) + \left(1-p\right)P\left(E \mid s_2\left(t\right)\right) \\ &= p\int_k^{\infty} \frac{\exp\left[-\left(v-s_{01}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}} dv + \left(1-p\right)\int_{-\infty}^{k} \frac{\exp\left[-\left(v-s_{02}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}} dv \end{aligned}$$

Use Leibnitz's rule to differentiate with respect to the threshold, k, set the result of the differentiation equal to 0, and solve for $k = k_{\text{opt}}$. The differentiation gives

$$-p\frac{\exp\left[-\left(k-s_{01}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}} + \left(1-p\right)\frac{\exp\left[-\left(k-s_{02}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}}\Bigg|_{k=k_{\text{opt}}} = 0$$

or

$$p\frac{\exp\left[-\left(k_{\text{opt}}-s_{01}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}}=(1-p)\frac{\exp\left[-\left(k_{\text{opt}}-s_{02}\right)^2/2\sigma_0^2\right]}{\sqrt{2\pi\sigma_0^2}}$$

or

$$\begin{aligned}-\left(k_{\text{opt}}-s_{01}\right)^2+\left(k_{\text{opt}}-s_{02}\right)^2 &= 2\sigma_0^2\ln\left(\frac{1-p}{p}\right)\\ \left(s_{01}-s_{02}\right)k_{\text{opt}}+\frac{s_{02}^2-s_{01}^2}{2} &= \sigma_0^2\ln\left(\frac{1-p}{p}\right)\\ k_{\text{opt}} &= \frac{\sigma_0^2}{s_{01}-s_{02}}\ln\left(\frac{1-p}{p}\right)+\frac{s_{02}-s_{01}}{2}\end{aligned}$$

Problem 7.11

(a) Denote the output due to the signal as $s_0(t)$ and the output due to noise as $n_0(t)$. We wish to maximize

$$\frac{s_0^2(t)}{E\left[n_0^2(t)\right]}=\frac{2}{N_0}\frac{\left|\int_{-\infty}^{\infty}S(f)H_m(f)e^{j2\pi ft_0}df\right|^2}{\int_{-\infty}^{\infty}H_m(f)H_m^*(f)df}$$

By Schwartz's inequality, we have

$$\frac{s_0^2(t)}{E\left[n_0^2(t)\right]}\leq\frac{2}{N_0}\frac{\int_{-\infty}^{\infty}S(f)S^*(f)df\int_{-\infty}^{\infty}H_m(f)H_m^*(f)df}{\int_{-\infty}^{\infty}H_m(f)H_m^*(f)e^{j2\pi ft_0}df}=\frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2df$$

where the maximum value on the right-hand side is attained if

$$H_m(f)=S^*(f)e^{-j2\pi ft_0}$$

(b) The matched filter impulse response is $h_m(t)=s(t_0-t)$ by taking the inverse Fourier transform of $H_m(f)$ employing the time reversal and time delay theorems.
(c) The realizable matched filter has zero impulse response for $t<0$.
(d) By sketching the components of the integrand of the convolution integral, we find the following results:

For $t_0=0,\ s_0(t_0)=0$;
For $t_0=T/2,\ s_0(t_0)=A^2T/2$;
For $t_0=T,\ s_0(t_0)=A^2T$;
For $t_0=2T,\ s_0(t_0)=A^2T$.

Problem 7.12
(a) These would be $x(t)$ and $y(t)$ reversed and then shifted to the right so that they are nonzero for $t > 0$.
(b) $A = \sqrt{7}B$.
(c) The outputs for the two cases are

$$\begin{aligned} S_{o1}(t) &= A^2\tau\Lambda\left[(t-t_0)/\tau\right] = 7B^2\tau\Lambda\left[(t-t_0)/\tau\right] \\ S_{o2}(t) &= 7B^2\tau\Lambda\left[(t-t_0)/7\tau\right] \end{aligned}$$

(d) The shorter pulse gives the more accurate time delay measurement.
(e) Peak power is lower for $y(t)$.

Problem 7.13
(a) The matched filter impulse response is given by

$$h_m(t) = s_2(T-t) - s_1(T-t)$$

(b) Using (7.55) and a sketch of $s_2(t) - s_1(t)$ for an arbitrary t_0, we obtain

$$\begin{aligned} \zeta^2 &= \frac{2}{N_0}\int_{-\infty}^{\infty}\left[s_2(t) - s_1(t)\right]^2 dt \\ &= \frac{2}{N_0}\left[A^2t_0 + 4A^2t_0 + A^2\left(\frac{T}{2} - t_0\right)\right] \\ &= \frac{2}{N_0}\left[\frac{A^2T}{2} + 4A^2t_0\right],\ 0 \le t_0 \le \frac{T}{2} \end{aligned}$$

This result increases linearly with t_0 and has its maximum value for for $t_0 = T/2$, which is $5A^2T/N_0$.
(c) The probability of error is

$$P_E = Q\left[\frac{\zeta}{2\sqrt{2}}\right]$$

To minimize it, use the maximum value of ζ, which is given in (b).
(d) The optimum receiver would have two correlators in parallel, one for $s_1(t)$ and one for $s_2(t)$. The outputs for the correlators at $t = T$ are differenced and compared with the threshold k calculated from (7.30), assuming the signals are equally likely.

Problem 7.14
Use $\zeta^2_{\max} = 2E_s/N_0$ for a matched filter, where E_s is the signal energy. The required values for signal energy are: (a) A^2T; (b) $3A^2T/8$; (c) $A^2T/2$; (d) $A^2T/3$ (Note that $A\Lambda\left[\frac{(t-T/2)}{T}\right]$ in the problem statement should be $A\Lambda\left[\frac{2(t-T/2)}{T}\right]$).

Problem 7.15
The required formulas are

$$\begin{aligned} E &= \frac{1}{2}\left(E_1 + E_2\right) \\ R_{12} &= \frac{1}{E}\int_0^T s_1\left(t\right) s_2\left(t\right) dt \\ k_{\text{opt}} &= \frac{1}{2}\left(E_2 - E_1\right) \\ P_E &= Q\left[\sqrt{\frac{\left(1 - R_{12}\right) E}{N_0}}\right] \end{aligned}$$

The energies for the three given signals are

$$E_A = A^2 T;\ E_B = B^2 T/2;\ E_C = 3C^2 T/8$$

(a) For $s_1 = s_A$ and $s_2 = s_B$, the average signal energy is

$$E = \frac{A^2 T}{2}\left(1 + \frac{B^2}{2A^2}\right)$$

The correlation coefficient is

$$\begin{aligned} R_{12} &= \frac{1}{E}\int_0^T AB\cos\left[\frac{\pi\left(t - T/2\right)}{T}\right] dt \\ &= \frac{1}{E}\int_0^T AB\sin\left[\frac{\pi t}{T}\right] dt = \frac{2ABT}{\pi E} \\ &= \frac{4AB}{\pi\left(A^2 + B^2/2\right)} \end{aligned}$$

The optimum threshold is

$$k_{\text{opt}} = \frac{1}{2}\left(E_B - E_A\right) = \frac{1}{2}\left(\frac{B^2}{2} - A^2\right) T$$

The probability of error is

$$P_E = Q\left[\sqrt{\frac{T}{2N_0}\left(A^2 + \frac{B^2}{2} - \frac{4AB}{\pi}\right)}\right]$$

(b) $s_1 = s_A$ and $s_2 = s_C$:

$$\begin{aligned} E &= \frac{1}{2}\left(E_A + E_C\right) = \frac{1}{2}\left(A^2 + 3C^2/8\right)T \\ R_{12} &= \frac{AC}{A^2 + 3C^2/8} \end{aligned}$$

$$k_{\text{opt}} = \frac{1}{2}\left(E_C - E_A\right) = \frac{1}{2}\left(\frac{3C^2}{8} - A^2\right)T$$

$$P_E = Q\left[\sqrt{\frac{T}{2N_0}\left(A^2 + \frac{3C^2}{8} - AC\right)}\right]$$

(c) $s_1 = s_B$ and $s_2 = s_C$:

$$\begin{aligned} E &= \frac{1}{2}\left(E_B + E_C\right) = \frac{1}{4}\left(B^2 + \frac{3C^2}{4}\right)T \\ R_{12} &= \frac{16BC}{3\pi\left(B^2 + 3C^2/4\right)} \end{aligned}$$

$$k_{\text{opt}} = \frac{1}{2}\left(E_C - E_B\right) = \frac{1}{4}\left(\frac{3C^2}{4} - B^2\right)T$$

$$P_E = Q\left[\sqrt{\frac{T}{4N_0}\left(B^2 + \frac{3C^2}{4} - \frac{16BC}{\pi}\right)}\right]$$

(d) $s_1 = s_B$ and $s_2 = -s_B$:

$$\begin{aligned} E &= E_B = B^2T/2 \\ R_{12} &= -1 \end{aligned}$$

$$k_{\text{opt}} = 0$$

$$P_E = Q\left[\sqrt{\frac{B^2T}{N_0}}\right]$$

(e) $s_1 = s_C$ and $s_2 = -s_C$:

$$\begin{aligned} E &= E_C = 3C^2T/8 \\ R_{12} &= -1 \end{aligned}$$

$$k_{\text{opt}} = 0$$

$$P_E = Q\left[\sqrt{\frac{3C^2T}{8N_0}}\right]$$

Problem 7.16
The appropriate equations to solve are

$$\begin{aligned} P_E &= Q\left[\sqrt{z}\right] = 10^{-5}, \text{ ASK} \\ P_E &= Q\left[\sqrt{2\left(1-m^2\right)z}\right] = 10^{-5}, \text{ PSK, } m = 0.2 \\ P_E &= Q\left[\sqrt{z}\right] = 10^{-5}, \text{ FSK} \end{aligned}$$

Trial and error shows that the Q-function has a value of 10^{-5} for it argument equal to 4.265. Thus, we get the following results:

For ASK and FSK, $\sqrt{z} = 4.265$ or $z = 18.19$ or $z_{\text{dB}} = 12.6$ dB
For PSK with $m = 0.2$, $\sqrt{2\left(1-m^2\right)z} = 4.265$ or $z = 9.47$ or $z_{\text{dB}} = 9.765$ dB

Problem 7.17
The program is given below. Curves corresponding to the numbers given in Table 7.2 are shown in Figure 7.2.

```
% Program bep_ph_error.m; Uses subprogram pb_phase_pdf
% bep_ph_error.m calls user-defined subprogram pb_phase_pdf.m
% Effect of Gaussian phase error in BPSK; fixed variance
clf
sigma_psi2 = input('Enter vector of variances of phase error in radians^2 ');
L_sig = length(sigma_psi2);
Eb_N0_dB_min = input('Enter minimum Eb/N0 desired ');
Eb_N0_dB_max = input('Enter maximum Eb/N0 desired ');
A = char('-.',':','–','-..');
a_mod = 0;
for m = 1:L_sig
```

```
    Eb_N0_dB = [];
    Eb_N0 = [];
    PE = [];
    k = 1
    for snr = Eb_N0_dB_min:.5:Eb_N0_dB_max
        Eb_N0_dB(k) = snr
        Eb_N0(k) = 10.^(Eb_N0_dB(k)/10);
        sigma_psi = sqrt(sigma_psi2(m));
        Eb_N0
        PE(k) = quad8('pb_phase_pdf',-pi,pi,[],[],Eb_N0(k),sigma_psi,a_mod)
        k = k+1;
    end
    PE
    semilogy(Eb_N0_dB, PE,A(m,:)),xlabel('E_b/N_0, dB'),ylabel('P_b'),...
    axis([Eb_N0_dB_min Eb_N0_dB_max 10^(-7) 1]),...
    if m == 1
        hold on
        grid on
    end
end
P_ideal = .5*erfc(sqrt(Eb_N0));
semilogy(Eb_N0_dB, P_ideal),...
legend(['BPSK; \sigma_\theta_e^2 = ',num2str(sigma_psi2(1))],
['BPSK; \sigma_\theta_e^2 = ',num2str(sigma_psi2(2))],
['BPSK; \sigma_\theta_e^2 = ',num2str(sigma_psi2(3))],
['BPSK; \sigma_\theta_e^2 = 0'],3)

% pb_phase_pdf.m; called by bep_ph_error.m and bep_ph_error2.m
%
function XX = pb_phase_pdf(psi,Eb_N0,sigma_psi,a)
arg = Eb_N0*(1-a^2)*(cos(psi)-a/sqrt(1-a^2)*sin(psi)).^2;
T1 = .5*erfc(sqrt(arg));
T2 = exp(-psi.^2/(2*sigma_psi^2))/sqrt(2*pi*sigma_psi^2);
XX = T1.*T2;
```

Problem 7.18

From the figure of Problem 7.17, the degradation for $\sigma_\phi^2 = 0.01$ is about 0.045 dB; for $\sigma_\phi^2 = 0.05$ it is about 0.6 dB; for $\sigma_\phi^2 = 0.1$ it is about 9.5 dB. For the constant phase error

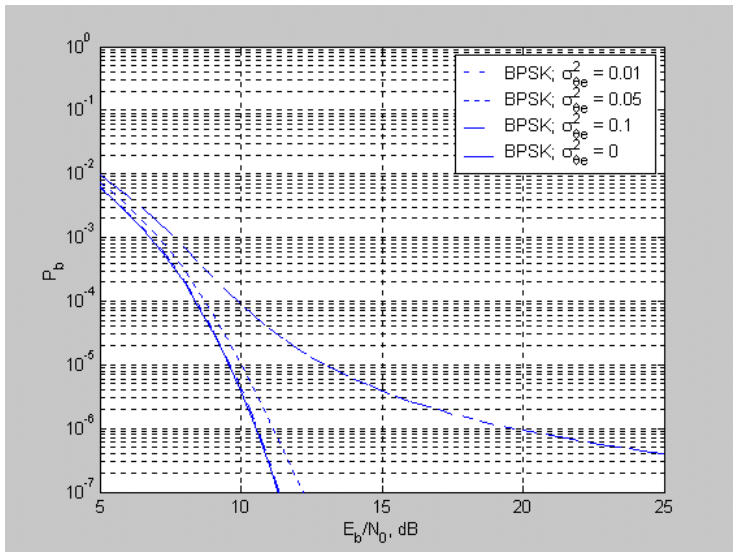

Figure 7.2:

model, the degradation is

$$D_{\text{const}} = -20 \log_{10} \left[\cos\left(\phi_{\text{const}}\right)\right]$$

As suggested in the problem statement, we set the constant phase error equal to the standard deviation of the Gaussian phase error for comparison putposes; thus, we consider constant phase errors of $\phi_{\text{const}} = 0.1,\ 0.224$, and 0.316 radians. The degradations corresponding to these phase errors are 0.044, 0.219, and 0.441 dB, respectively. The constant phase error degradations are much less serious than the Gaussian phase error degradations.

Problem 7.19

(a) For ASK, $P_E = Q\left[\sqrt{z}\right] = 10^{-5}$ gives $z = 18.19$ or $z = 12.6$ dB.
(b) For BPSK, $P_E = Q\left[\sqrt{2z}\right] = 10^{-5}$ gives $z = 9.1$ or $z = 9.59$ dB.
(c) Binary FSK is the same as ASK.
(d) The degradation of BPSK with a phase error of 5 degrees is $D_{\text{const}} = -20 \log_{10}\left[\cos\left(5^{\text{o}}\right)\right] = 0.033$ dB, so the required SNR to give a bit error probability of 10^{-5} is $9.59 + 0.033 = 9.623$ dB.
(e) For PSK with $m = 1/\sqrt{2}$, $P_E = Q\left[\sqrt{2\left(1 - 1/2\right)z}\right] = Q\left[\sqrt{z}\right] = 10^{-5}$ gives $z = 18.19$ or $z = 12.6$ dB.
(f) Assuming that the separate effects are additive, we add the degradation of part (d) to the SNR found in part (e) to get $12.6 + 0.033 = 12.633$ dB.

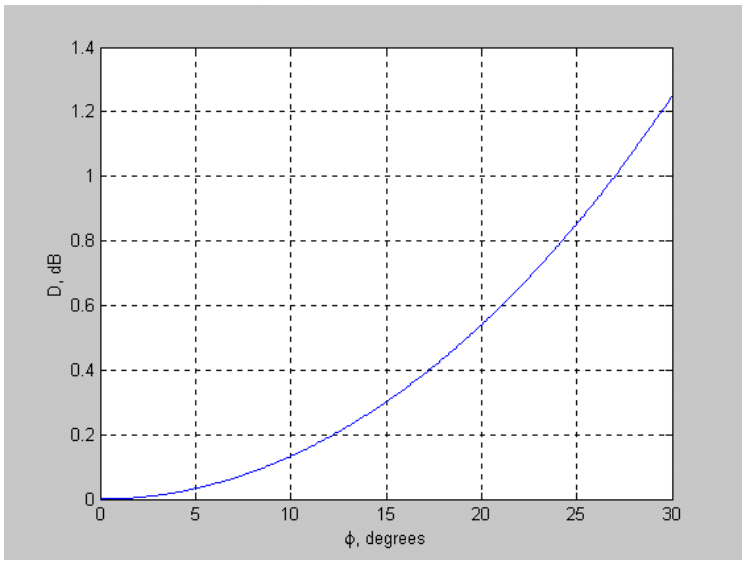

Figure 7.3:

Problem 7.20
The degradation in dB is $D_{\text{const}} = -20\log_{10}[\cos(\phi)]$. It is plotted in Figure 7.3.
The degradations for phase errors of 3, 5, 10, and 15 degrees are 0.0119, 0.0331, 0.1330, and 0.3011 dB, respectively.

Problem 7.21
(a) Solve $P_E = Q\left(\sqrt{2z}\right) = 10^{-4}$ to get $z = 8.4$ dB. From (7.73), the additional SNR required due to a carrier component is $-10\log_{10}\left(1-m^2\right)$. A plot of SNR versus m is given in Fig. 7.4.
(b) Solve $P_E = Q\left(\sqrt{z}\right) = 10^{-5}$ to get $z = 9.57$ dB. A plot of SNR versus m is given in Fig. 7.4.
(c) Solve $P_E = Q\left(\sqrt{z}\right) = 10^{-6}$ to get $z = 10.53$ dB. A plot of SNR versus m is given in Fig. 7.4.

Problem 7.22
(a) For BPSK, an SNR of 10.53 dB is required to give an error probability of 10^{-6}. For ASK and FSK it is 3 dB more, or 13.54 dB. For BPSK and ASK, the required bandwidth is $2R$ Hz where R is the data rate in bps. For FSK with minimum tone spacing of $0.5R$ Hz, the required bandwidth is $2.5R$ Hz. Hence, for BPSK and ASK, the required bandwidth is 200 kHz. For FSK, it is 250 kHz.

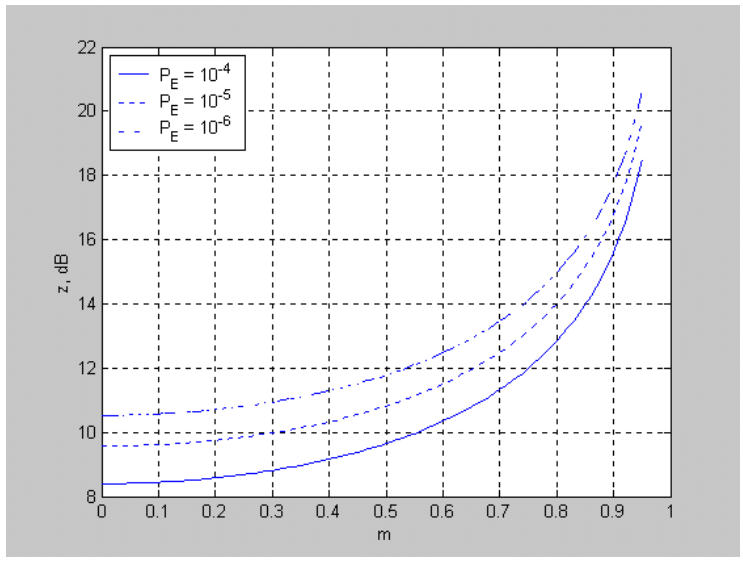

Figure 7.4:

(b) For BPSK, the required SNR is 9.6 dB. For ASK and FSK it is 12.6 dB. The required bandwidths are now 400 kHz for BPSK and ASK; it is 500 kHz for FSK.

Problem 7.23

The correlation coefficient is

$$\begin{aligned} R_{12} &= \frac{1}{E}\int_0^T s_1(t)\, s_2(t)\, dt \\ &= \frac{1}{E}\int_0^T A^2 \cos(\omega_c t) \cos[(\omega_c + \Delta\omega)\, t]\, dt \\ &= \operatorname{sinc}(2\Delta f T) \end{aligned}$$

where Δf is the frequency separation between the FSK signals in Hz. Using a calculator, we find that the sinc-function takes on its first minimum at an argument value of about 1.4, or $\Delta f = 0.7/T$. The minimum value at this argument value is about -0.216. The improvement in SNR is $-10\log_{10}(1 - R_{12}) = -10\log_{10}(1 + 0.216) = 0.85$ dB.

Problem 7.24

The encoded bit streams, assuming a reference 1 to start with, are: (a) 1 100 001 000 010; (b) 1 100 110 011 001; (c) 1 111 111 111 111; (d)1 010 101 010 101; (e) 1 111 111 010 101; (f) 1 110 000 011 011; (g) 1 111 010 000 101; (h) 1 100 001 000 010.

Problem 7.25
The encoded bit stream is 1 110 000 100 110 (1 assumed for the starting reference bit). The phase of the transmitted carrier is $000\pi\pi\pi\pi0\pi\pi00\pi$. Integrate the product of the received signal and a 1-bit delayed signal over a $(0,\ T)$ interval. Each time the integrated product is greater than 0, decide 1; each time it is less than 0, decide 0. The resulting decisions are 110 111 001 010, which is the message signal. If the signal is inverted, the same demodulated signal is detected as if it weren't inverted.

Problem 7.26
Note that

$$E\left[n_1\right] = E\left[\int_{-T}^{0} n\left(t\right)\cos\left(\omega_c t\right)dt\right] = \int_{-T}^{0} E\left[n\left(t\right)\right]\cos\left(\omega_c t\right)dt = 0$$

which follows because $E\left[n\left(t\right)\right] = 0$. Similarly for n_2, n_3, and n_4. Consider the variance of each of these random variables:

$$\begin{aligned}
\mathrm{var}\left[n_1\right] &= E\left[n_1^2\right] = E\left[\int_{-T}^{0}\int_{-T}^{0} n\left(t\right)n\left(\lambda\right)\cos\left(\omega_c t\right)\cos\left(\omega_c \lambda\right)dtd\lambda\right] \\
&= \int_{-T}^{0}\int_{-T}^{0} E\left[n\left(t\right)n\left(\lambda\right)\right]\cos\left(\omega_c t\right)\cos\left(\omega_c \lambda\right)dtd\lambda \\
&= \int_{-T}^{0}\int_{-T}^{0} \frac{N_0}{2}\delta\left(t-\lambda\right)\cos\left(\omega_c t\right)\cos\left(\omega_c \lambda\right)dtd\lambda \\
&= \frac{N_0}{2}\int_{-T}^{0}\cos^2\left(\omega_c t\right)dt = \frac{N_0 T}{4}
\end{aligned}$$

where the fact that $E\left[n\left(t\right)n\left(\lambda\right)\right] = \frac{N_0}{2}\delta\left(t-\lambda\right)$ has been used along with the sifting property of the δ-function to reduce the double integral to a single integral. Similarly for n_2, n_3, and n_4. Therefore, w_1 has zero mean and variance

$$\mathrm{var}\left[w_1\right] = \frac{1}{4}\mathrm{var}\left[n_1\right] + \frac{1}{4}\mathrm{var}\left[n_2\right] = \frac{N_0 T}{8}$$

Similarly for w_2, w_3, and w_4.

Problem 7.27
The two error probabilities are

$$P_{E,\ \mathrm{opt}} = \frac{1}{2}e^{-z} \text{ and } P_{E,\ \text{delay-and-mult}} \approx Q\left(\sqrt{z}\right)$$

Comparative values are:

z, dB	$P_{E,\text{ opt}}$	$P_{E,\text{ delay-and-mult}}$
8	9.1×10^{-4}	6.8×10^{-3}
9	1.8×10^{-4}	2.7×10^{-3}
10	2.3×10^{-5}	8.5×10^{-4}
11	1.7×10^{-6}	2.1×10^{-4}

Problem 7.28
Noncoherent binary FSK has error probability

$$P_{E,\text{ NCFSK}} = \frac{1}{2}e^{-z_{\text{NCFSK}}/2}$$

Solve for the SNR to get

$$z_{\text{NCFSK}} = -2\ln\left(2P_{E,\text{ NCFSK}}\right)$$

DPSK has error probability

$$P_{E,\text{ DPSK}} = \frac{1}{2}e^{-z_{\text{DPSK}}}$$

Solve for the SNR to get

$$z_{\text{DPSK}} = -\ln\left(2P_{E,\text{ DPSK}}\right)$$

The values asked for are given in the table below:

P_E	z_{NCFSK}, dB	z_{DPSK}, dB
10^{-3}	10.94	7.93
10^{-5}	13.35	10.34
10^{-7}	14.89	11.88

Problem 7.29
Bandwidths are given by

$$\begin{aligned}
B_{\text{BPSK}} &= B_{\text{DPSK}} = 2R;\ R_{\text{BPSK}} = R_{\text{BPSK}} = B_{\text{DPSK}}/2 = 50 \text{ kbps} \\
B_{\text{CFSK}} &= 2.5R;\ R_{\text{CFSK}} = B_{\text{BPSK}}/2.5 = 40 \text{ kbps} \\
B_{\text{NCFSK}} &= 4R;\ R_{\text{NCFSK}} = B_{\text{BPSK}}/4 = 25 \text{ kbps}
\end{aligned}$$

Problem 7.30

Consider the bandpass filter, just wide enough to pass the ASK signal, followed by an envelope detector with a sampler and threshold comparator at its output. The output of the bandpass filter is

$$\begin{aligned} u(t) &= A_k \cos(\omega_c t + \Theta) + n(t) \\ &= A_k \cos(\omega_c t + \Theta) + n_c(t) \cos(\omega_c t + \Theta) - n_s(t) \sin(\omega_c t + \Theta) \\ &= x(t) \cos(\omega_c t + \Theta) - n_s(t) \sin(\omega_c t + \Theta) \end{aligned}$$

where $A_k = A$ if signal plus noise is present at the receiver input, and $A = 0$ if noise alone is present at the input. In the last equation

$$x(t) = A_k + n_c(t)$$

Now $u(t)$ can be written in envelope-phase form as

$$u(t) = r(t) \cos[\omega_c t + \Theta + \phi(t)]$$

where

$$r(t) = \sqrt{x^2(t) + n_s^2(t)} \text{ and } \phi(t) = \tan^{-1}\left[\frac{n_s(t)}{x(t)}\right]$$

The output of the envelope detector is $r(t)$, which is sampled each T seconds and compared with the threshold. Thus, to compute the error probability, we need the pdf of the envelope for noise alone at the input and then with signal plus noise at the input. For noise alone, it was shown in Chapter 4 that the pdf of $r(t)$ is Rayleigh, which is

$$f_R(r) = \frac{r}{R} e^{-r^2/2N}, \ r \geq 0, \text{ noise only}$$

where N is the variance (mean-square value) of the noise at the filter output. For signal plus noise at the receiver input, the pdf of the envelope is Ricean. Rather than use this complex expression, however, we observe that for large SNR at the receiver input

$$r(t) \approx A + n_c(t), \text{ large SNR, signal present}$$

which follows by expanding the argument of the square root, dropping the squared terms, and approximating the square root as

$$\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon, \ |\epsilon| << 1$$

Thus, for large SNR conditions, the envelope detector output is approximately Gaussian for signal plus noise present. It has mean A and variance N (recall that the inphase and

quadrature lowpass noise components have variance the same as the bandpass process). The pdf of the envelope detector output with signal plus noise present is approximately

$$f_R(r) = \frac{e^{-(r-A)^2/2N}}{\sqrt{2\pi N}}, \text{ large SNR}$$

For large SNR, the threshold is approximately $A/2$. Thus, for noise alone present, the probability of error is exactly

$$P(E \mid 0) = \int_{A/2}^{\infty} \frac{r}{R} e^{-r^2/2N} dr = e^{-A^2/8N}$$

For signal plus noise present, the probability of error is approximately

$$\begin{aligned} P(E \mid S+N \text{ present}) &= \int_{-\infty}^{A/2} \frac{e^{-(r-A)^2/2N}}{\sqrt{2\pi N}} dr = Q\left(\sqrt{A^2/2N}\right) \\ &\approx \frac{e^{-A^2/4N}}{\sqrt{\pi/N}A} \end{aligned}$$

Now the average signal-to-noise ratio is

$$z = \frac{1}{2}\frac{A^2/2}{N_0 B_T} = \frac{A^2}{4N}$$

which follows because the signal is present only half the time. Therefore, the average probability of error is

$$P_E = \frac{1}{2}P(E \mid S+N) + \frac{1}{2}P(E \mid 0) \approx \frac{e^{-z}}{\sqrt{4\pi z}} + \frac{1}{2}e^{-z/2}$$

Problem 7.31

This is a matter of following the steps as outlined in the problem statement.

Problem 7.32

Do the inverse Fourier transform of the given $P(f)$ to show that $p(t)$ results. The derivation is quite long and requires a fair amount of trigonometry. Only a sketch is given here. It is also useful to note that for an even function, the inverse Fourier transform integral reduces to

$$p(t) = \int_0^{\infty} P(f) \exp(j2\pi ft)\, df = 2\int_0^{\infty} P(f) \cos(2\pi ft)\, df$$

because $P(f)$ is even. Now substitute for $P(f)$ from (7.124) to get

$$
\begin{aligned}
p(t) &= 2\int_0^{\frac{1-\beta}{2T}} T\cos(2\pi f t)\,df + 2\int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{\beta}\left(f-\frac{1-\beta}{2T}\right)\right]\right\}\cos(2\pi f t)\,df \\
&= T\frac{\sin\left[2\pi\left(\frac{1-\beta}{2T}\right)t\right]}{2\pi t} + T\frac{\sin\left[2\pi\left(\frac{1+\beta}{2T}\right)t\right]}{2\pi t} \\
&\quad +\frac{T}{2}\int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}}\left\{\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}+\frac{\pi}{2}\right]+\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}+\frac{\pi}{2}\right]\right\}df \\
&= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\
&\quad -\frac{T}{2}\int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}}\left\{\sin\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}\right]+\sin\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}\right]\right\}df \\
&= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\
&\quad +\frac{T}{2}\left[\frac{\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}+\frac{\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)f-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right]_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \\
&= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\
&\quad +\frac{T}{2}\left[\frac{\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)\frac{1+\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}+\frac{\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)\frac{1+\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right] \\
&\quad -\frac{T}{2}\left[\frac{\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)\frac{1-\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}+\frac{\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)\frac{1-\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right] \\
&= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\
&\quad +\frac{T}{2}\left[\frac{\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)\frac{1+\beta}{2T}-\frac{\pi}{2\beta}\right]-\cos\left[\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)\frac{1-\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}\right] \\
&\quad +\frac{T}{2}\left[\frac{\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)\frac{1+\beta}{2T}-\frac{\pi}{2\beta}\right]-\cos\left[\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)\frac{1-\beta}{2T}-\frac{\pi}{2\beta}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right]
\end{aligned}
$$

Expand the trigonometric function arguments to get

$$\begin{aligned} p(t) &= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\ &\quad + \frac{T}{2}\left[\frac{\cos\left[(1+\beta)\frac{\pi t}{T}+\frac{\pi}{2}\right]-\cos\left[(1-\beta)\frac{\pi t}{T}-\frac{\pi}{2}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}\right] \\ &\quad + \frac{T}{2}\left[\frac{\cos\left[(1+\beta)\frac{\pi t}{T}-\frac{\pi}{2}\right]-\cos\left[(1-\beta)\frac{\pi t}{T}+\frac{\pi}{2}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right] \end{aligned}$$

Expand the cosines using to get

$$\begin{aligned} p(t) &= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T) \\ &\quad -\frac{T}{2}\left[\frac{\sin\left[(1+\beta)\frac{\pi t}{T}\right]+\sin\left[(1-\beta)\frac{\pi t}{T}\right]}{\frac{\pi T}{\beta}\left(1+\frac{2\beta}{T}t\right)}\right]+\frac{T}{2}\left[\frac{\sin\left[(1+\beta)\frac{\pi t}{T}\right]+\sin\left[(1-\beta)\frac{\pi t}{T}\right]}{\frac{\pi T}{\beta}\left(1-\frac{2\beta}{T}t\right)}\right] \\ &= \cos(\pi\beta t/T)\ \operatorname{sinc}(t/T)-\left[\frac{\cos(\pi\beta t/T)\sin(\pi t/T)}{\frac{2\pi t}{T}\left(\frac{T}{2\beta t}+1\right)}\right]+\left[\frac{\cos(\pi\beta t/T)\sin(\pi t/T)}{\frac{2\pi t}{T}\left(\frac{T}{2\beta t}-1\right)}\right] \\ &= \frac{T}{\pi t}\left[1-\frac{1}{2\left(\frac{T}{2\beta t}+1\right)}+\frac{1}{2\left(\frac{T}{2\beta t}-1\right)}\right]\cos(\pi\beta t/T)\sin(\pi t/T) \\ &= \frac{\cos(\pi\beta t/T)\ \operatorname{sinc}(t/T)}{1-(2\beta t/T)^2} \end{aligned}$$

Problem 7.33
For $\beta = 0.25$,

$$P(f)=\begin{cases} T,\ 0\le|f|\le\frac{0.75}{2T} \\ \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{0.25}\left(|f|-\frac{0.75}{2T}\right)\right]\right\},\ \frac{0.75}{2T}\le|f|\le\frac{1.25}{2T}\ \text{Hz} \\ 0,\ |f|>\frac{0.75}{2T} \end{cases}$$

The optimum transmitter and receiver transfer functions are

$$|H_T(f)|_{\text{opt}}=\frac{A\,|P(f)|^{1/2}\,G_n^{1/4}(f)}{|H_C(f)|^{1/2}}$$

and

$$|H_R(f)|_{\text{opt}} = \frac{K\,|P(f)|^{1/2}}{G_n^{1/4}(f)\,|H_C(f)|^{1/2}}$$

Setting all scale factors equal to 1, these become

$$|H_T(f)|_{\text{opt}} = \frac{|P(f)|^{1/2}\ \sqrt[1/4]{1+(f/f_C)^2}}{\sqrt[1/4]{1+(f/f_3)^2}}$$

$$|H_R(f)|_{\text{opt}} = |P(f)|^{1/2}\ \sqrt[1/4]{1+(f/f_3)^2}\ \sqrt[1/4]{1+(f/f_C)^2}$$

A MATL:AB program for computing the optimum transfer functions is given below. Plots for the various cases are then given in Figures 7.5 - 7.7.

```
% Problem 7.33: Optimum transmit and receive filters for raised-cosine pulse shaping,
% first-order Butterworth channel filtering, and first-order noise spectrum
%
A = char('-',':','–','-.');
clf
beta00 = input('Enter starting value for beta: >=0 and <= 1 ');
del_beta = input('Enter step in beta: starting value + 4*step <= 1 ');
R = 1;
f_3 = R/2;
f_C = R/2;
delf = 0.01;
HT_opt = [];
HR_opt = [];
for n = 1:4
    beta = (n-1)*del_beta+beta00;
    beta0(n) = beta;
    f1 = 0:delf:(1-beta)*R/2;
    f2 = (1-beta)*R/2:delf:(1+beta)*R/2;
    f3 = (1+beta)*R/2:delf:R;
    f = [f1 f2 f3];
    P1 = ones(size(f1));
    P2 = 0.5*(1+cos(pi/(R*beta)*(f2-0.5*(1-beta)*R)));
    P3 = zeros(size(f3));
    Gn = 1./(1+(f/f_3).^2);
```

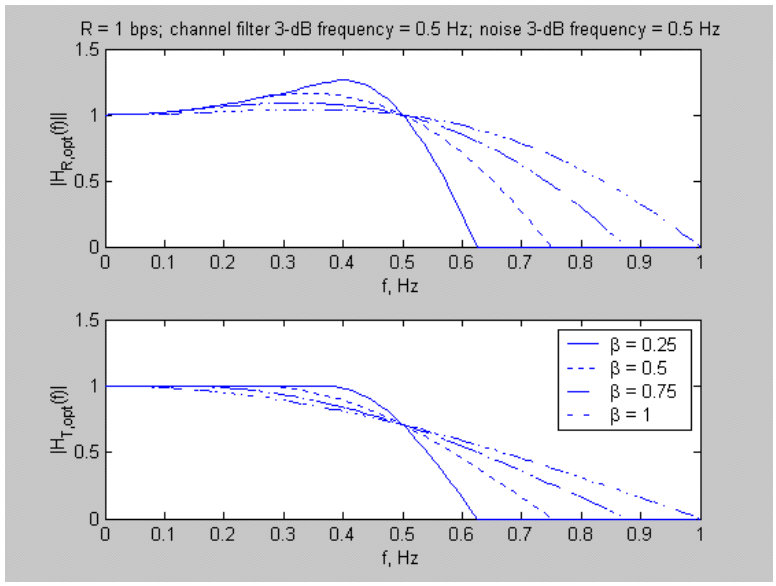

Figure 7.5: $f_3 = f_C = 1/2T$

```
        HC = 1./sqrt(1+(f/f_C).^2);
        P = [P1 P2 P3];
        HT_opt = sqrt(P).*(Gn.^.25)./sqrt(HC);
        HR_opt = sqrt(P)./((Gn.^.25).*sqrt(HC));
        subplot(2,1,1),plot(f,HR_opt,A(n,:)),xlabel('f, Hz'),ylabel('|H_R_,_o_p_t(f)||')
        title(['R = ',num2str(R),' bps; channel filter 3-dB frequency = ',num2str(f_C),'
            Hz; noise 3-dB frequency = ',num2str(f_3),' Hz'])
        if n == 1
            hold on
        end
        subplot(2,1,2),plot(f,HT_opt,A(n,:)),xlabel('f, Hz'),ylabel('|H_T_,_o_p_t(f)|')
        if n == 1
            hold on
        end
    end
    legend(['\beta = ',num2str(beta0(1))],['\beta = ',num2str(beta0(2))],
        ['\beta = ',num2str(beta0(3))],['\beta = ',num2str(beta0(4))],1)
```

(a) $f_3 = f_C = 1/2T$

(b) $f_C = 2f_3 = 1/T$

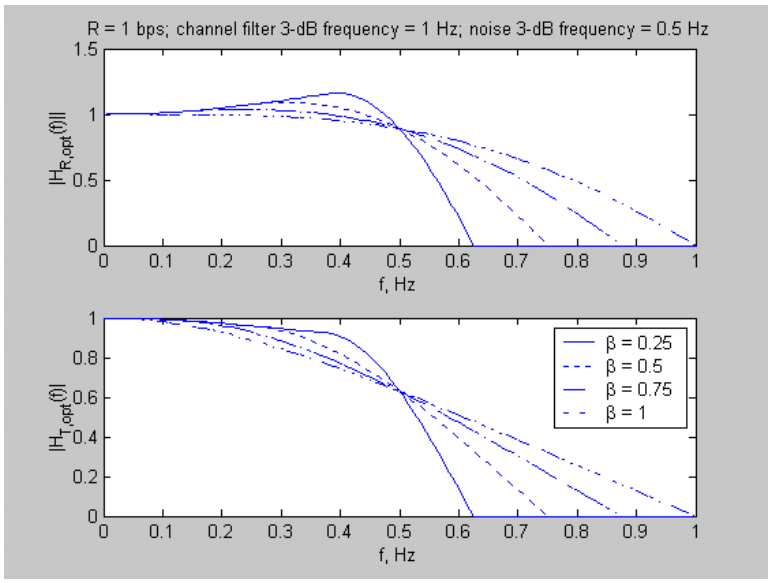

Figure 7.6: $f_C = 2f_3 = 1/T$

(c) $f_3 = 2f_C = 1/T$

Problem 7.34

This a simple matter of sketching $\Lambda(f/B)$ and $\Lambda\left[\left(f - \frac{1}{2}B\right)/B\right]$ side by side and showing that they add to a constant in the overlap region.

Problem 7.35

Follow the steps outlined in the problem statement.

Problem 7.36

(a) The optimum transmitter and receiver transfer functions are

$$|H_T(f)|_{\text{opt}} = \frac{A\,|P(f)|^{1/2}\,G_n^{1/4}(f)}{|H_C(f)|^{1/2}}$$

and

$$|H_R(f)|_{\text{opt}} = \frac{K\,|P(f)|^{1/2}}{G_n^{1/4}(f)\,|H_C(f)|^{1/2}}$$

where $P(f)$ is the signal spectrum and A and K are arbitrary constants which we set equal

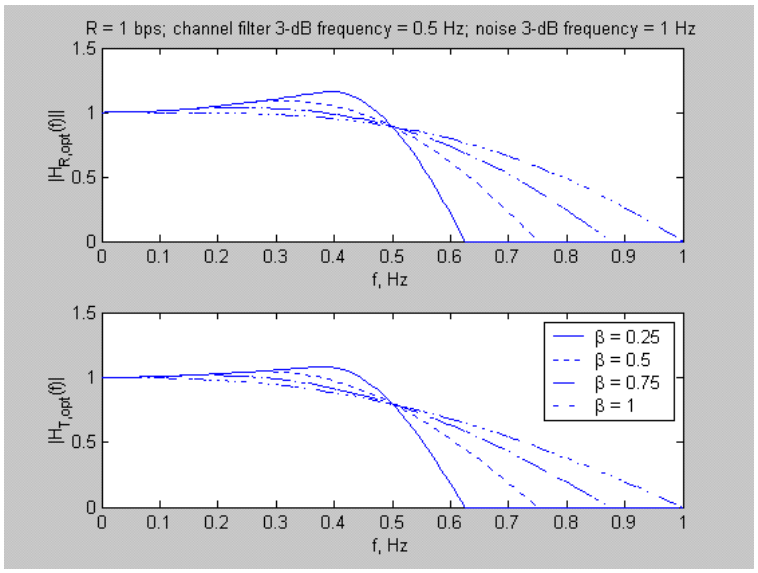

Figure 7.7: $f_3 = 2f_C = 1/T$

to 1 for convenience. From the problem statement, $G_n^{1/4}(f) = \left(10^{-12}\right)^{1/4} = 10^{-3}$ and

$$|H_C(f)| = \frac{1}{\sqrt{1 + (f/4800)^2}}$$

The signal spectrum is, for $\beta = 1$,

$$\begin{aligned} P(f) &= \frac{1}{2 \times 4800}\left[1 + \cos\left(\frac{\pi |f|}{4800}\right)\right],\ 0 \le |f| \le 4800 \text{ Hz} \\ &= \frac{1}{4800}\cos^2\left(\frac{\pi |f|}{9600}\right),\ 0 \le |f| \le 4800 \text{ Hz} \end{aligned}$$

Thus,

$$|H_T(f)|_{\text{opt}} = |H_R(f)|_{\text{opt}} = \left|\cos\left(\frac{\pi |f|}{9600}\right)\right| \sqrt{1 + (f/4800)^2},\ 0 \le |f| \le 4800 \text{ Hz}$$

where all scale factors have been set equal to 1.

Problem 7.37

The MATLAB program is given below and Figure 7.8 shows the probability of error curves.

```
% Plots for Prob. 7.37
%
clf
A = char('-','–','-.',':','–.','-..');
delta = input(' Enter value for delta: ')
taum_T0 = input(' Enter vector of values for tau_m/T: ')
L_taum = length(taum_T0);
z0_dB = 0:.1:15;
z0 = 10.^(z0_dB/10);
for ll = 1:L_taum
    ll
    taum_T = taum_T0(ll)
    P_E = 0.5*qfn(sqrt(2*z0)*(1+delta))+0.5*qfn(sqrt(2*z0)*((1+delta)-2*delta*taum_T));
    semilogy(z0_dB, P_E,A(ll,:))
    if ll == 1
        hold on; grid on; axis([0, inf, 10^(-6), 1]); xlabel('z_0, dB'); ylabel('P_E')
    end
end
title(['P_E versus z_0 in dB for \delta = ', num2str(delta)])
if L_taum == 1
    legend(['\tau_m/T = ',num2str(taum_T0)],3)
elseif L_taum == 2
    legend(['\tau_m/T = ',num2str(taum_T0(1))],['\tau_m/T = ',num2str(taum_T0(2))],3)
elseif L_taum == 3
    legend(['\tau_m/T = ',num2str(taum_T0(1))],['\tau_m/T = ',num2str(taum_T0(2))],
    ['\tau_m/T = ',num2str(taum_T0(3))],3)
elseif L_taum == 4
    legend(['\tau_m/T = ',num2str(taum_T0(1))],['\tau_m/T = ',num2str(taum_T0(2))],
    ['\tau_m/T = ',num2str(taum_T0(3))],['\tau_m/T = ',num2str(taum_T0(4))],3)
end
```

Problem 7.38

The program given in Problem 7.37 can be adapted for making the asked for plot.

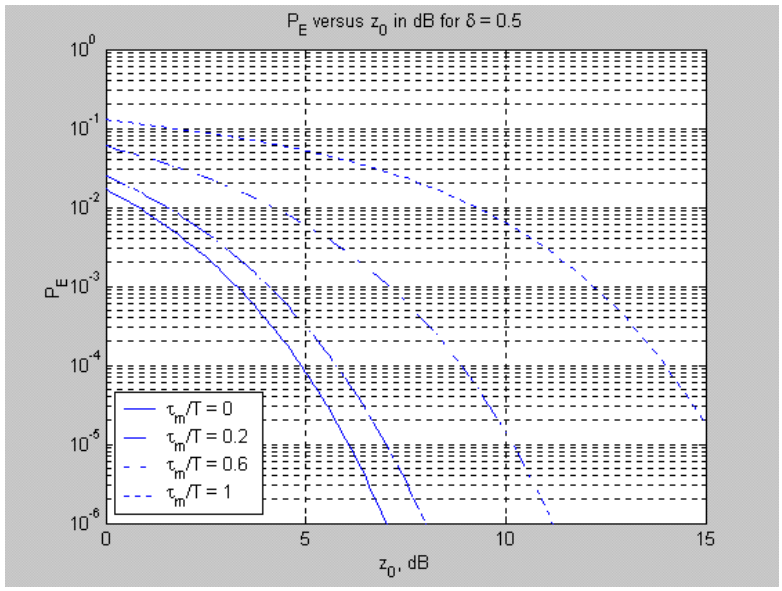

Figure 7.8:

Problem 7.39

(a) The output of the channel in terms of the input is

$$y(t) = x(t) + \beta x(t - \tau_m)$$

Fourier transform both sides and take the ratio of output to input transforms to obtain the channel transfer function:

$$H_C(f) = \frac{Y(f)}{X(f)} = 1 + \beta e^{-j2\pi f \tau_m}$$

Using Euler's theorem to expand the exponential and finding the maginitude gives

$$|H_C(f)| = \sqrt{1 + 2\beta \cos(2\pi f \tau_m) + \beta^2}$$

Plots are shown in Fig. 7.9 for $\beta = 0.5$ and 1.0.

```
% Plots for Prob. 7.39
%
clf
A = char('-','–','-.',':','–.','-..');
tau_m = input(' Enter value for tau_m: ')
```

```
beta0 = input(' Enter vector of values for beta: ')
L_beta = length(beta0)
f = -1:.005:1;
for ll = 1:L_beta
    beta = beta0(ll);
    arg = 1+2*beta*cos(2*pi*tau_m*f)+beta^2;
    mag_H = sqrt(arg);
    plot(f, mag_H, A(ll,:))
    if ll == 1
        hold on; grid on; xlabel('f'); ylabel('|H(f)|')
    end
end
title(['|H(f)| versus f for \tau_m = ', num2str(tau_m)])
if L_beta == 1
    legend(['\beta = ',num2str(beta0)],3)
elseif L_beta == 2
    legend(['\beta = ',num2str(beta0(1))],['\beta = ',num2str(beta0(2))],3)
elseif L_beta == 3
    legend(['\beta = ',num2str(beta0(1))],['\beta = ',num2str(beta0(2))],
    ['\beta = ',num2str(beta0(3))],3)
elseif L_beta == 4
    legend(['\beta = ',num2str(beta0(1))],['\beta = ',num2str(beta0(2))],
    ['\beta = ',num2str(beta0(3))],['\beta = ',num2str(beta0(4))],3)
end
```

(b) The output of the tapped delay line filter in terms of the input is

$$z(t) = \sum_{n=0}^{N} \beta_n y\left[t - (n-1)\Delta\right]$$

Fourier transform this equation and take the ratio of the output to the input transforms to obtain the transfer function of the equalizer as

$$H_{\text{eq}}(f) = \frac{Z(f)}{Y(f)} = \sum_{n=0}^{\infty} \beta^n e^{-j2\pi(n-1)\Delta f}$$

(c) From part (a) and using the geometric expansion, we obtain

$$\frac{1}{H_C(f)} = \frac{1}{1+\beta e^{-j2\pi f \tau_m}} = \sum_{n=0}^{\infty} (-1)^n \beta^n e^{-j2\pi n \tau_m f}$$

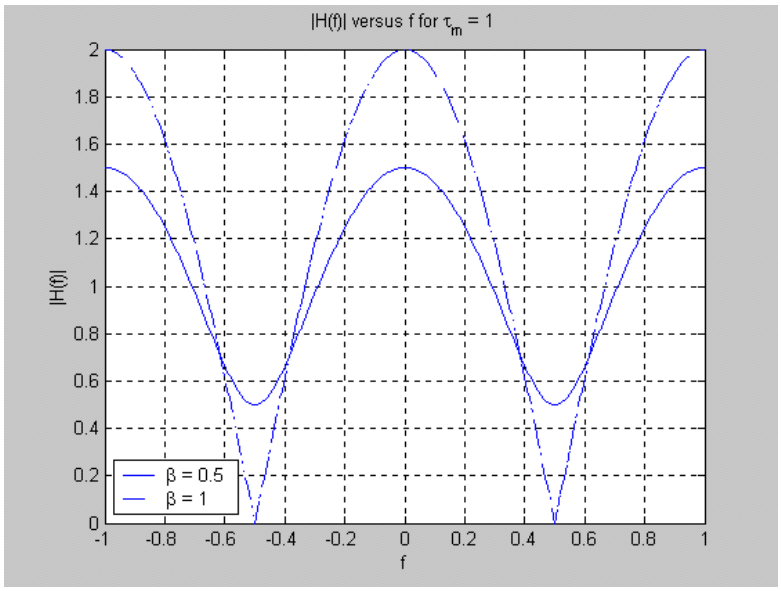

Figure 7.9:

Equate this to the result of part (b) with $\tau_m = \Delta$. Clearly, $\beta_n = (-\beta)^{n-1}$.

Problem 7.40

The appropriate equations are:

BPSK: (7.73) with $m = 0$ for nonfading; (7.174) for fading;

DPSK: (7.111) for nonfading; (7.176) for fading;

CFSK: (7.87) for nonfading; (7.175) for fading;

NFSK: (7.119) for nonfading; (7.177) for fading.

Degradations for $P_E = 10^{-2}$ are:

Modulation type	Fading SNR, dB	Nonfaded SNR, dB	Fading Margin, dB
BPSK	13.85	4.32	9.52
DPSK	16.90	5.92	10.98
Coh. FSK	16.86	7.33	9.52
Noncoh. FSK	19.91	8.93	10.98

Problem 7.41

See Problem 7.40 for equation reference numbers. The inverse equations, expressing signal-to-noise ratio in dB, are:

(a) BPSK

$$\begin{aligned} z_{\text{BPSK, NF}} &= 10\log_{10}\left\{\frac{1}{2}\left[Q^{-1}\left(P_E\right)\right]^2\right\} \text{ dB} \\ z_{\text{BPSK, F}} &= 10\log_{10}\left\{\frac{\left(1-2P_E\right)^2}{1-\left(1-2P_E\right)^2}\right\} \text{ dB} \end{aligned}$$

The margin in dB is the difference between these two results for a given P_E.
(b) DPSK

$$\begin{aligned} z_{\text{DPSK, NF}} &= 10\log_{10}\left\{-\ln\left(2P_E\right)\right\} \text{ dB} \\ z_{\text{DPSK, F}} &= 10\log_{10}\left\{\frac{1}{2P_E}-1\right\} \text{ dB} \end{aligned}$$

(c) Coherent FSK

$$\begin{aligned} z_{\text{BPSK, NF}} &= 10\log_{10}\left\{\left[Q^{-1}\left(P_E\right)\right]^2\right\} \text{ dB} \\ z_{\text{BPSK, F}} &= 10\log_{10}\left\{\frac{2\left(1-2P_E\right)^2}{1-\left(1-2P_E\right)^2}\right\} \text{ dB} \end{aligned}$$

(d) Noncoherent FSK

$$\begin{aligned} z_{\text{NCFSK, NF}} &= 10\log_{10}\left\{-2\ln\left(2P_E\right)\right\} \text{ dB} \\ z_{\text{NCFSK, F}} &= 10\log_{10}\left\{\frac{1}{P_E}-2\right\} \text{ dB} \end{aligned}$$

The fading margin for these various modulation cases are given in the table below for $P_E = 10^{-3}$.

Modulation type	Fading SNR, dB	Nonfaded SNR, dB	Fading Margin, dB
BPSK	23.97	6.79	17.18
DPSK	26.98	7.93	19.05
Coh. FSK	26.98	9.80	17.18
Noncoh. FSK	29.99	10.94	19.05

Problem 7.42
The expressions with Q-functions can be integrated by parts (BPSK and coherent FSK). The other two cases involve exponential integrals which are easily integrated (DPSK and noncoherent FSK).

Problem 7.43
We need to invert the matrix

$$[P_C] = \begin{bmatrix} 1 & 0.1 & -0.01 \\ 0.2 & 1 & 0.1 \\ -0.02 & 0.2 & 1 \end{bmatrix}$$

Using a calculator or MATLAB, we find the inverse matrix to be

$$[P_C]^{-1} = \begin{bmatrix} 1.02 & -0.11 & 0.02 \\ -0.21 & 1.04 & -0.11 \\ 0.06 & -0.21 & 1.02 \end{bmatrix}$$

The optimum weights are the middle column.
(a) The equation giving the equalized output is

$$p_{\text{eq}}(mT) = -0.11 p_C[(m+1)T] + 1.04 p_C[mT] - 0.21 p_C[(m-1)T]$$

(b) Calculations with the above equation using the given sample values result in

$$p_{\text{eq}}(-2T) = -0.02;\ p_{\text{eq}}(-T) = 0;\ p_{\text{eq}}(0) = 1;\ p_{\text{eq}}(T) = 0;\ p_{\text{eq}}(2T) = -0.06$$

Note that equalized values two samples away from the center sample are not zero because the equalizer can only provide three specified values.

Problem 7.44
(a) The desired matrix is

$$[R_{yy}] = \frac{N_0}{T} \begin{bmatrix} \left(1+b^2\right) z + \frac{\pi}{2} & bz + \frac{\pi}{2} e^{-2\pi} & \frac{\pi}{2} e^{-4\pi} \\ bz + \frac{\pi}{2} e^{-2\pi} & \left(1+b^2\right) z + \frac{\pi}{2} & bz + \frac{\pi}{2} e^{-2\pi} \\ \frac{\pi}{2} e^{-4\pi} & bz + \frac{\pi}{2} e^{-2\pi} & \left(1+b^2\right) z + \frac{\pi}{2} \end{bmatrix}$$

(b) The following matrix is needed:

$$[R_{yd}] = \begin{bmatrix} R_{yd}(-T) \\ R_{yd}(0) \\ R_{yd}(T) \end{bmatrix} = \begin{bmatrix} 0 \\ A \\ bA \end{bmatrix}$$

Multiply the matrix given in (a) by the column matrix of unknown weights to get the equations

$$\frac{N_0}{T} \begin{bmatrix} \left(1+b^2\right) z + \frac{\pi}{2} & bz + \frac{\pi}{2} e^{-2\pi} & \frac{\pi}{2} e^{-4\pi} \\ bz + \frac{\pi}{2} e^{-2\pi} & \left(1+b^2\right) z + \frac{\pi}{2} & bz + \frac{\pi}{2} e^{-2\pi} \\ \frac{\pi}{2} e^{-4\pi} & bz + \frac{\pi}{2} e^{-2\pi} & \left(1+b^2\right) z + \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} a'_{-1} \\ a'_0 \\ a'_1 \end{bmatrix} = \begin{bmatrix} 0 \\ A \\ bA \end{bmatrix}$$

or

$$\begin{bmatrix} \left(1+b^2\right)z+\frac{\pi}{2} & bz+\frac{\pi}{2}e^{-2\pi} & \frac{\pi}{2}e^{-4\pi} \\ bz+\frac{\pi}{2}e^{-2\pi} & \left(1+b^2\right)z+\frac{\pi}{2} & bz+\frac{\pi}{2}e^{-2\pi} \\ \frac{\pi}{2}e^{-4\pi} & bz+\frac{\pi}{2}e^{-2\pi} & \left(1+b^2\right)z+\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ A^2T/N_0 \\ bA^2T/N_0 \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ bz \end{bmatrix}$$

where the factor N_0/T has been normalized out so that the right hand side is in terms of the signal-to-noise ratio, z (the extra factor of A needed mulitplies both sides of the matrix equation and on the left hand side is lumped into the new coefficients a_{-1}, a_0, and a_1. Invert the matrix equation above to find the weights. A MATLAB program for doing so is given below. For $z = 10$ dB and $b = 0.1$, the weights are $a_{-1} = -0.2781$, $a_0 = 0.7821$, and $a_1 = 0.0773$.

```
% Solution for Problem 7.44
%
z_dB = input('Enter the signal-to-noise ratio in dB ');
z = 10^(z_dB/10);
b = input('Enter multipath gain ');
R_yy(1,1)=(1+b^2)*z+pi/2;
R_yy(2,2)=R_yy(1,1);
R_yy(3,3)=R_yy(1,1);
R_yy(1,2)=b*z+(pi/2)*exp(-2*pi);
R_yy(1,3)=(pi/2)*exp(-4*pi);
R_yy(3,1)=R_yy(1,3);
R_yy(2,1)=R_yy(1,2);
R_yy(2,3)=R_yy(1,2);
R_yy(3,2)=R_yy(1,2);
disp('R_yy')
disp(R_yy)
B = inv(R_yy);
disp('R_yy^-1')
disp(B)
R_yd = [0 z b*z]'
A = B*R_yd
```

Typical output of the program:

```
>> pr7_44
Enter the signal-to-noise ratio in dB 10
Enter multipath gain .5
R_yy                                    R_yd =
```

```
        14.0708 5.0029 0.0000                    0
        5.0029 14.0708 5.0029                    10
        0.0000 5.0029 14.0708                    5
    R_yy^-1                                      A =
        0.0831 -0.0338 0.0120                   -0.2781
        -0.0338 0.0951 -0.0338                   0.7821
        0.0120 -0.0338 0.0831                    0.0773
```

7.2 Computer Exercises

Computer Exercise 7.1

```
% ce7_1.m: Simulation BEP for antipodal signaling
    %
    clf
    n_sim = input('Enter number of bits for simulation: ');
    zdB = input('Enter z = A^2*T/N_0 vector in dB: ');
    z = 10.^(zdB/10);
    Lz = length(z);
    A = 1;
    T = 1;
    N0 = (A^2*T)./z;
    S = A*T*sign(rand(1,n_sim)-.5);
    PE = [];
    ET = [];
    for k = 1:Lz
        N = sqrt(0.5*N0(k)*T)*randn(1,n_sim);
        Y = S+N;
        Z = sign(Y);
        V = 0.5*(1+Z);
        W = 0.5*(1+sign(S));
        E = xor(V,W);
        ET(k) = sum(E);
        PE(k) = ET(k)/length(Y);
    end
    PEth = 0.5*erfc(sqrt(z));
    disp(' ')
    disp(' SNR, dB      Errors      P_E est.      P_E theory')
    disp(' ____________________________________________')
```

```
disp(' ')
disp([zdB' ET' PE' PEth'])
disp(' ')
semilogy(zdB, PE, 'o'), xlabel('z in dB'), ylabel('P_E'), grid
hold on
semilogy(zdB, PEth)
title(['BER simulated with ', num2str(n_sim), ' bits per SNR value '])
legend(['Simulated'], ['Theoretical'])
```

A typical run is given below:

```
>> ce7_1
Enter number of bits for simulation: 20000
Enter z = A^2*T/N_0 vector in dB: [2 4 6 8]
SNR, dB      Errors      P_E est.      P_E theory
-----------------------------------------
2.0000      704.0000      0.0352      0.0375
4.0000      238.0000      0.0119      0.0125
6.0000       44.0000      0.0022      0.0024
8.0000        4.0000      0.0002      0.0002
```

A plot also appears showing the estimated and theoretical probabilities of error plotted versus signal-to-noise ratio.

Computer Exercise 7.2

```
% ce7_2: Compute degradation in SNR due to bit timing error
%
clf
A = char('-','-.',':','–');
for m = 1:4
PE0 = 10^(-m);
delT = [];
deg = [];
Eb1 = [];
k = 0;
for Y = 0:.025:.45;
    k = k+1;
    delT(k) = Y;
    Eb1(k) = fzero(@PE1, 2, [], delT(k), PE0);
end
Eb = fzero(@PE, 2, [], PE0);
```

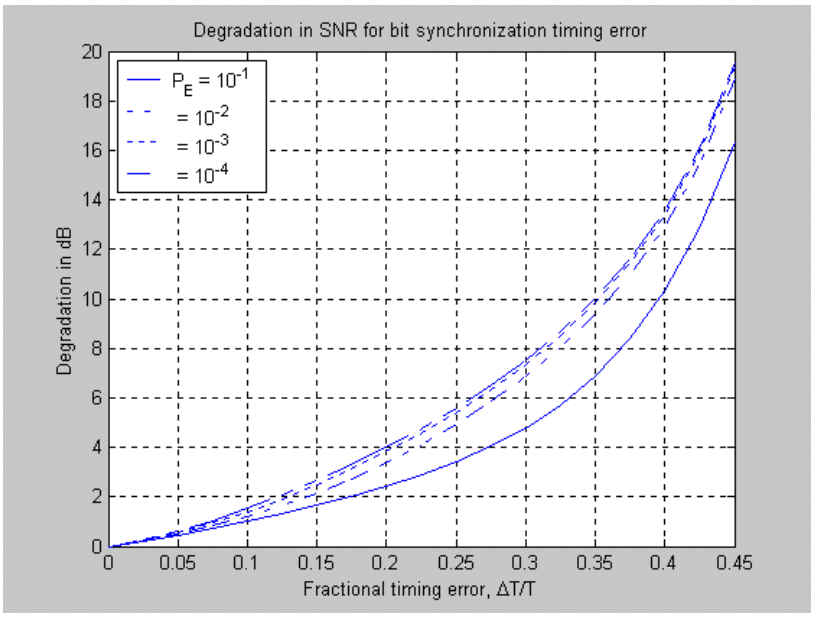

Figure 7.10:

```
deg = Eb1 - Eb;
plot(delT, deg, A(m,:)), ...
if m == 1
hold on
axis([0 .45 0 20])
xlabel('Fractional timing error, \DeltaT/T'), ylabel('Degradation in dB'),...
title(['Degradation in SNR for bit synchronization timing error']), grid
end
end
legend(['P_E = 10^-^1'], [' = 10^-^2'], [' = 10^-^3'], [' = 10^-^4'], 2)
```

The output plot is shown in Fig. 7.1.

Computer Exercise 7.3

A MATLAB program for this computer exercise is given in the solution for Problem 7.17.

Computer Exercise 7.4

```
% ce7_4: For a given data rate and error probability, find the required
%       bandwidth and Eb/N0 in dB; baseband, BPSK/DPSK, coherent FSK
%       and noncoherent FSK modulation may be specified.
%
I_mod = input('Enter desired modulation: 0 = baseband; 1 = BPSK; 2 = DPSK;
                    3 = CFSK; 4 = NCFSK: ');
I_BW_R = input('Enter 1 if bandwidth specified; 2 if data rate specified: ');
if I_BW_R == 1
    BkHz = input('Enter vector of desired bandwidths in kHz: ');
elseif I_BW_R == 2
    Rkbps = input('Enter vector of desired data rates in kbps: ');
end
PE = input('Enter desired BEP: ');
N0 = 1;
if I_BW_R == 1
    if I_mod == 0
        Rkpbs = BkHz;          % Rate for baseband
    elseif I_mod == 1 | I_mod == 2
        Rkbps = BkHz/2;        % Rate in kbps for BPSK or DPSK
    elseif I_mod == 3
        Rkbps = BkHz/3;        % Rate in kbps for CFSK
    elseif I_mod == 4
        Rkbps = BkHz/4;        % Rate in kbps for NFSK
    end
elseif I_BW_R == 2
    if I_mod == 0
        BkHz = Rkbps;           % Transmission BW for baseband in kHz
    elseif I_mod == 1 | I_mod == 2
        BkHz = 2*Rkbps;        % Transmission BW for BPSK or DPSK in kHz
    elseif I_mod == 3
        BkHz = 3*Rkbps;        % Transmission BW for CFSK in kHz
    elseif I_mod == 4
        BkHz = 4*Rkbps;        % Transmission BW for NFSK in kHz
    end
end
R = Rkbps*1000;
B = BkHz*1000;
% PE = 0.5*erfc(sqrt(z)) MATLAB has an inverse erf but not an inverse Q-function
```

```
if I_mod == 0 | I_mod == 1
    sqrt_z = erfinv(1-2*PE);
    z = sqrt_z^2;
elseif I_mod == 2
    z = -log(2*PE);
elseif I_mod == 3
    sqrt_z_over_2 = erfinv(1-2*PE);
    z = 2*sqrt_z_over_2^2;
elseif I_mod == 4
    z = -2*log(2*PE);
end
Eb_N0_dB = 10*log10(z);      % This is the required Eb/N0 in dB
if I_mod == 0
    A = sqrt(R*N0*z);          % z = A^2/(R*N0) for baseband
elseif I_mod == 1 | I_mod == 2 | I_mod == 3 | I_mod == 4
    A = sqrt(2*R*N0*z);      % z = A^2/2*R*N0 for BPSK, DPSK, CFSK, NFSK
end
disp(' ')
disp('Desired P_E and required E_b/N_0 in dB: ')
format long
disp([PE Eb_N0_dB])
format short
disp(' ')
disp(' R, kbps BW, Hz A, volts')
disp(' ____________________________')
disp(' ')
disp([Rkbps' BkHz' A'])
disp(' ')
```

A typical run is given below:

```
>> ce7_4
Enter desired modulation: 0 = baseband; 1 = BPSK; 2 = DPSK;
                                        3 = CFSK; 4 = NCFSK: 3
Enter 1 if bandwidth specified; 2 if data rate specified: 2
Enter vector of desired data rates in kbps: [5 10 20 50]
Enter desired BEP: 1e-3
Desired P_E and required E_b/N_0 in dB:
    0.00100000000000 9.79982256904398
R, kbps      BW, Hz      A, volts
-----------------------------------
  5.0000      15.0000     309.0232
 10.0000      30.0000     437.0248
 20.0000      60.0000     618.0465
 50.0000     150.0000     977.2173
```

Computer Exercise 7.5

```
% ce7_5.m: Simulation of suboptimum bandpass filter/delay-and-multiply
% demodulator with integrate-and-dump detection for DPSK; input bandpass
% filter is Butterworth with selectable bandwidth-bit period product.
% Simulation is broken up into contiguous blocks for memory management.
%
Eb_N0_dB_max = input('Enter maximum Eb/N0 in dB: ');
Eb_N0_dB_min = input('Enter minimum Eb/N0 in dB: ');
samp_bit = input('Enter number of samples per bit used in simulation: ');
n_order = input('Enter order of Butterworth detection filter: ');
BWT_bit = input('Enter filter bandwidth normalized by bit rate: ');
N_bits = input('Enter total number of bits in simulation: ');
N_bits_block = input('Enter number of bits/sim. block: ');
N_blocks = floor(N_bits/N_bits_block);
T_bit = 1; % Arbitrarily take bit time as 1 second
BW = BWT_bit/T_bit;                % Compute filter bandwidth from BW*T_bit
[num,den] = butter(n_order,2*BW/samp_bit);  % Obtain filter num/den coefficients
e_tot = [];
N_bits_sim = [];
Eb_N0_dB = [];                      % Ensure that plotting arrays are empty
Perror = [];
clf                                % Clear any plots left over from previous runs
k = 0;
```

```
for kk = Eb_N0_dB_min:Eb_N0_dB_max % Loop for simul. det. for each Eb/N0
    k = k+1;
    Eb_N0_dB(k) = kk;
    Eb_N0 = 10^(Eb_N0_dB(k)/10);     % Convert desired Eb/N0 from dB to ratio
    Eb = T_bit;                       % Bit energy is T_bit, if ampl. = 1
    N0 = Eb/Eb_N0;                    % Compute noise PSD from Eb/N0
    del_t = T_bit/samp_bit;           % Compute sampling interval
    sigma_n = sqrt(N0/(2*del_t));     % Compute std deviations of noise samples
    e_sum_tot = 0;
    zi = [];
    nn = 1;
    while nn <= N_blocks & e_sum_tot < 200
        ss = sign(rand(1,N_bits_block)-.5); % Generate sequence of random +-1s
        data = 0.5*(ss+1);            % Logical data is sequence of 1s and 0s
        data_diff_enc = diff_enc(data);   % Differentially encode data for DPSK
        s = 2*data_diff_enc-1;        % Generate bipolar data for modulation
        sig = [];
        sig = s(ones(samp_bit,1),:);  % Build array wth columns samp_bit long
        sig = sig(:);                 % Convert matrix of bit samples to vector
        bits_out = [];                % Make sure various arrays are empty
        y_det = [];
        noise = sigma_n*randn(size(sig)); % Form sequence of Gauss. noise samples
        [y,zf] = filter(num,den,sig+noise,zi); % Filter S plus N with chosen filter
        zi = zf;                      % Save final values for future init. cond'ns
        y_ref = delay1(y,samp_bit);   % Ref. signal is 1-bit delayed S + Ne
        y_mult = y.*y_ref;            % Multiply rec'd S + N by 1 bit delay
        bits_out = int_and_dump(y_mult,samp_bit,N_bits_block); % I-&-D det.
        error_array = abs(bits_out-data); % Compare detected bits with input
        if nn == 1
            error_array(1:10) = 0;    % Don't include 1st 5 bits due to transients
        elseif nn ~= 1
            error_array(1:1) = 0;     % Delete 1st bit due to y_ref = 0
        end
        e_sum = sum(error_array);     % Sum to get total number of errors/blk
        e_sum_tot = e_sum_tot + e_sum; % Running sum of total no. of errors
        nn = nn + 1;
    end
    e_tot(k) = e_sum_tot;
    N_bits_sim(k) = (nn-1)*N_bits_block-10-(nn-2);
    Perror(k) = e_sum_tot/N_bits_sim(k);
```

```
end                                     % End of Eb/N0 loop
disp(' ')
disp(' Eb/N0, dB; PE errors')       % Display computed prob. of error
disp(' ____________________________')
disp(' ')
disp([Eb_N0_dB' Perror' e_tot'])
disp(' ')
% Plot simulated bit error probabilities versus Eb/N0
semilogy(Eb_N0_dB, Perror,'-'), grid, xlabel('E_b/N_0; dB'), ylabel('P_E'), hold
% Plot theoretical bit error probability for optimum DPSK detector
semilogy(Eb_N0_dB, 0.5*exp(-10.^(Eb_N0_dB/10)))
% Plot approximate theoretical result for suboptimum detector
semilogy(Eb_N0_dB, qfn(sqrt(10.^(Eb_N0_dB/10))),'-.')
Title(['Comparison of optimum & delay-and-multiply detectors for DPSK; ',
      num2str(n_order), '-order Butterworth filter; BT = ', num2str(BWT_bit)])
legend(['Simulation; BT = ', num2str(BWT_bit),'; ', num2str(N_bits),' bits; ',
      num2str(N_blocks), ' blocks'],'Theory; optimum differential detector',
     'Theory; delay/multiply detector', 3)

%diff_enc(input); function to differentially encode a bit stream vector
%
function output = diff_enc(input)
L_in = length(input);
output = [];
for k = 1:L_in
if k == 1
output(k) = not(bitxor(input(k),1));
else
output(k) = not(bitxor(input(k),output(k-1)));
end
end

% int_and_dump(input,samp_bit,N_bits);
%
function bits_out = int_and_dump(input,samp_bit,N_bits)
% Reshape input vector with each bit occupying a column
samp_array = reshape(input, samp_bit, N_bits);
integrate = sum(samp_array); % Integrate (sum) each bit (column)
bits_out = (sign(integrate) + 1)/2;
```

A typical run is given below. A minimum of 200 errors are counted unless the total number of specified bits to be simulated is exceeded. A plot is also made comparing probabilities of error for the optimum detector, simulated suboptimum detector, and approximate theoretical result for the suboptimum detector.

```
>> ce7_5
Enter maximum Eb/N0 in dB: 7
Enter minimum Eb/N0 in dB: 3
Enter number of samples per bit used in simulation: 5
Enter order of Butterworth detection filter: 2
Enter filter bandwidth normalized by bit rate: 1.5
Enter total number of bits in simulation: 15000
Enter number of bits/sim. block: 5000
Eb/N0,      dB; PE      errors
---------------------------
3.0000      0.1040      519.0000
4.0000      0.0655      327.0000
5.0000      0.0485      242.0000
6.0000      0.0216      216.0000
7.0000      0.0101      152.0000
```

Computer Exercise 7.6

```
% ce7_6: program to plot Figure 7-28
%
clf
z0dB = 0:1:30;
z0 = 10.^(z0dB/10);
for delta = -0.9:0.3:0.6;
    if delta > -eps & delta < eps
        delta = 0;
    end
    for taum_over_T = 0:1:1
        T1 = 0.5*qfn(sqrt(2*z0)*(1+delta));
        T2 = 0.5*qfn(sqrt(2*z0)*(1+delta)-2*delta*taum_over_T);
        PE = T1 + T2;
        if taum_over_T == 0
            semilogy(z0dB, PE, '-')
        elseif taum_over_T == 1
            semilogy(z0dB, PE, '–')
        end
```

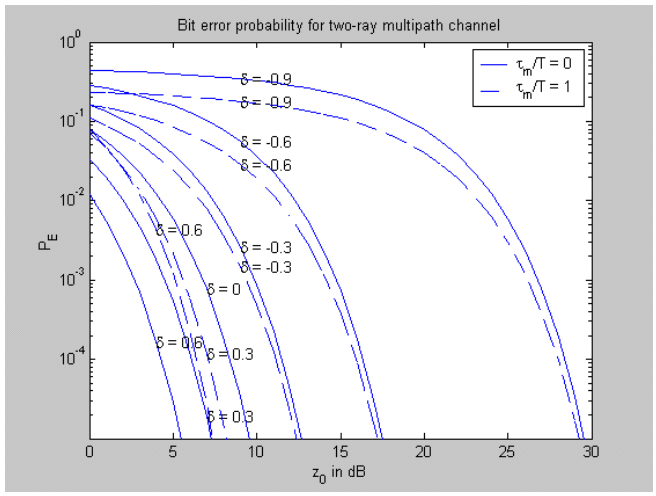

Figure 7.11:

```
            if delta <= -.3
                text(z0dB(10), PE(10), ['\delta = ', num2str(delta)])
            elseif delta > -0.3 & delta <= 0.3
                text(z0dB(8), PE(8), ['\delta = ', num2str(delta)])
            elseif delta > 0.3
                text(z0dB(5), PE(5), ['\delta = ', num2str(delta)])
            end
            if delta == -0.9
                hold on
                ylabel('P_E'), xlabel('z_0 in dB')
                axis([0 30 1E-5 1])
                grid
            end
        end
    end
    legend(['\tau_m/T = 0'],['\tau_m/T = 1'])
```

The output plot is shown in Fig. 7.2.

Computer Exercise 7.7

Use Equations (7.73) (with $m = 0$), (7.87), (7.111), (7.119), and (7.174) - (7.177). Solve them for the SNR in terms of the error probability. Express the SNR in dB and subtract

the nonfading result from the fading result. The MATLAB program below does this for the modulation cases of BPSK, coherent FSK (CFSK), DPSK, and noncoherent (NFSK).

```
% ce7_7.m: Program to evaluate degradation due to flat
% Rayleigh fading for various modulation schemes
%
mod_type = input('Enter type of modulation: 1 = BPSK; 2 = CFSK; 3 = DPSK; 4 =
NFSK: ');
PE = input('Enter vector of desired probabilities of error: ');
disp(' ')
if mod_type == 1
    disp(' BPSK')
    Z_bar = (0.25*(1 - 2*PE).^2)./(PE - PE.^2);
    z = (erfinv(1 - 2*PE)).^2; % MATLAB has an inv. error function
elseif mod_type == 2
    disp(' CFSK')
    Z_bar = (0.5*(1 - 2*PE).^2)./(PE - PE.^2);
    z = 2*(erfinv(1 - 2*PE)).^2;
elseif mod_type == 3
    disp(' DPSK')
    Z_bar = 1./(2*PE) -1;
    z = -log(2*PE);
elseif mod_type == 4
    disp(' NFSK')
    Z_bar = 1./PE -2;
    z = -2*log(2*PE);
end
Z_bar_dB = 10*log10(Z_bar);
z_dB = 10*log10(z);
deg_dB = Z_bar_dB - z_dB;
disp(' ')
disp(' P_E Degradation, dB')
disp('___________________________')
disp(' ')
disp([PE' deg_dB'])
disp(' ')
```

A typical run is given below:

```
>> ce7_7
Enter type of modulation: 1 = BPSK; 2 = CFSK; 3 = DPSK; 4 = NFSK: 4
```

```
Enter vector of desired probabilities of error: [1e-2 5e-3 1e-3 5e-4 1e-4]
            NFSK
    P_E       Degradation, dB
--------------------------
    0.0100      10.9779
    0.0050      13.3239
    0.0010      19.0469
    0.0005      21.6023
    0.0001      27.6859
```

Computer Exercise 7.8

The solution to this exercise consists of two MATLAB programs, one for the zero-forcing case and one for the MMSE case.

```
% ce7_8a.m: Plots unequalized and equalized sample values for ZF equalizer
%
clf
pc = input('Enter sample values for channel pulse response (odd #): ');
L_pc = length(pc);
disp('Maximum number of zeros each side of decision sample: ');
disp((L_pc-1)/4)
N = input('Enter number of zeros each side of decision sample desired: ');
mid_samp = fix(L_pc/2)+1;
Pc = [];
for m = 1:2*N+1
    row = [];
    for n = 2-m:2*N+2-m
        row = [row pc(mid_samp-1+n)];
    end
    Pc(:,m) = row';
end
disp(' ')
disp('Pc:')
disp(' ')
disp(Pc)
Pc_inv = inv(Pc);
disp(' ')
disp('Inverse of Pc:')
disp(' ')
disp(Pc_inv)
```

```
coef = Pc_inv(:,N+1);
disp(' ')
disp('Equalizer coefficients:')
disp(' ')
disp(coef)
disp(' ')
steps = (L_pc-length(coef))+1;
p_eq = [];
for m = mid_samp-floor(steps/2):mid_samp+floor(steps/2)
    p_eq(m) = sum(coef'.*fliplr(pc(m-N:m+N)));
end
p_eq_p = p_eq(N+1:mid_samp+floor(steps/2));
disp(' ')
disp('Equalized pulse train')
disp(' ')
disp(p_eq_p)
disp(' ')
t=-5:.01:5;
y=zeros(size(t));
subplot(2,1,1),stem(pc),ylabel('Ch. output pulse'),...
axis([1 L_pc -.5 1.5]),...
title(['Ch. pulse samp. = ',num2str(pc),']'])
subplot(2,1,2),stem(p_eq_p),xlabel('n'),ylabel('Equal. output pulse'),...
axis([1 L_pc -.5 1.5]),...
title(['Output equalized with ',num2str(N),' zero samples either side'])
```

A typical run is given below:

```
>> ce7_8a
Enter sample values for channel pulse response (odd #):
                    [-.05 .1 -.15 .25 -.3 1 -.2 .15 -.1 .05 -.02]
Maximum number of zeros each side of decision sample:
2.5000
Enter number of zeros each side of decision sample desired: 2
Pc:
1.0000     -0.3000     0.2500     -0.1500     0.1000
-0.2000     1.0000     -0.3000     0.2500     -0.1500
0.1500 -    0.2000     1.0000     -0.3000     0.2500
-0.1000     0.1500 -    0.2000     1.0000     -0.3000
0.0500 -    0.1000     0.1500     -0.2000     1.0000
Inverse of Pc:
```

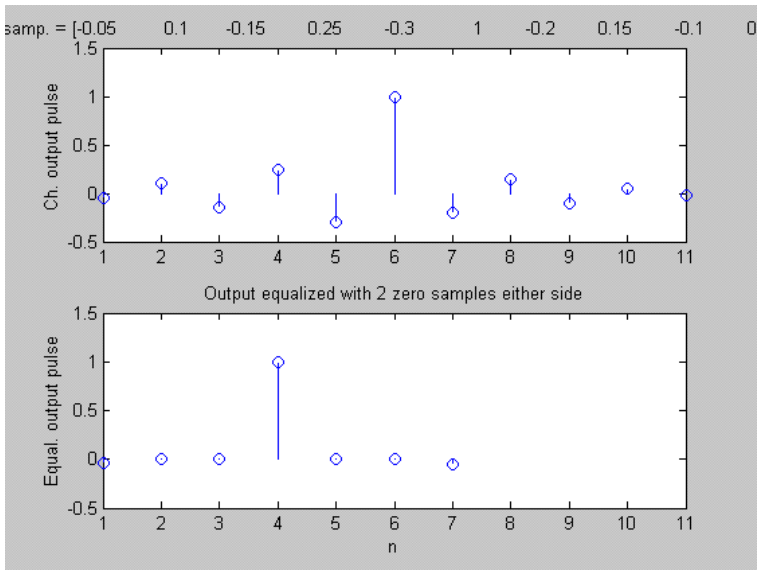

Figure 7.12:

1.0877	0.2842	-0.1777	0.0366	-0.0107
0.1686	1.1317	0.2572	-0.1731	0.0366
-0.1095	0.1403	1.1476	0.2572	-0.1777
0.0588	-0.0956	0.1403	1.1317	0.2842
-0.0093	0.0588	-0.1095	0.1686	1.0877

Equalizer coefficients:
-0.1777
0.2572
1.1476
0.1403
-0.1095
Equalized pulse train

-0.0350	-0.0000	-0.0000	1.0000	0.0000	0	-0.0554

The MATLAB program for the MMSE case is given below:

```
% Solution for ce 7.8(b)
%
z_dB = input('Enter the signal-to-noise ratio in dB: ');
z = 10^(z_dB/10);
b = input('Enter multipath gain: ');
```

```
R_yy(1,1)=(1+b^2)*z+pi/2;
R_yy(2,2)=R_yy(1,1);
R_yy(3,3)=R_yy(1,1);
R_yy(1,2)=b*z+(pi/2)*exp(-2*pi);
R_yy(1,3)=(pi/2)*exp(-4*pi);
R_yy(3,1)=R_yy(1,3);
R_yy(2,1)=R_yy(1,2);
R_yy(2,3)=R_yy(1,2);
R_yy(3,2)=R_yy(1,2);
disp(' ')
disp('R_yy:')
disp(' ')
disp(R_yy)
B = inv(R_yy);
disp(' ')
disp('R_yy^-1:')
disp(' ')
disp(B)
R_yd = [0 z b*z]';
disp(' ')
disp('R_yd:')
disp(' ')
disp(R_yd)
A = B*R_yd;
disp(' ')
disp('Optimum coefficients:')
disp(A)
disp(' ')
```

A typical run is given below:

```
>> ce7_8b
Enter the signal-to-noise ratio in dB: 7
Enter multipath gain: .6
R_yy:
8.3869      3.0101      0.0000
3.0101      8.3869      3.0101
0.0000      3.0101      8.3869
R_yy^-1:
0.1399      -0.0576      0.0207
-0.0576      0.1606     -0.0576
```

```
0.0207      -0.0576      0.1399
R_yd:
0
5.0119
3.0071
Optimum coefficients:
-0.2267
0.6316
0.1319
```

Chapter 8

Advanced Data Communications Topics

8.1 Problem Solutions

Problem 8.1
Use the relationship

$$R_b = (\log_2 M)\, R_s \text{ bps}$$

where R_b is the data rate, R_s is the symbol rate, and M is the number of possible signals per signaling interval. In this case, $R_s = 2000$ symbols per second. For $M = 4$, $R_b = 2 \times 2,000 = 4,000$ bps, for $M = 8$, $R_b = 6,000$ bps, and for $M = 64$, $R_b = 12,000$ bps.

Problem 8.2
(a) 5,000 symbols per second. (b) Recall that

$$y(t) = A\left[d_1(t)\cos(\omega_c t) + d_2(t)\sin(\omega_c t)\right] = \sqrt{2}A\cos\left[\omega_c t - \theta_i(t)\right],\ \theta_i(t) = \tan^{-1}\left[\frac{d_2(t)}{d_1(t)}\right]$$

Alternating every other bit given in the problem statement between $d_1(t)$ and $d_2(t)$ gives $d_1(t) = 1,\ 1, -1,\ 1,\ 1, -1,\ 1, -1$ and $d_2(t) = 1, -1, -1, -1, -1,\ 1, -1$ which results in $\theta_i(t) = \pi/4,\ 7\pi/4,\ 5\pi/4,\ 7\pi/4,\ 7\pi/4,\ 3\pi/4,\ 7\pi/4,\ \cdots$. For QPSK the symbol switching points occur each T_s seconds. (c) Now the switching instants are each T_b seconds. Start with $d_1(t) = 1$. Then $d_2(t)$ is staggered, or offset, by 1 bit time. So the first phase shift is for $d_1(t) = 1$ and $d_2(t) = 1$ or $\theta_2(t) = \pi/4$. After $T_s = 2T_b$ seconds $d_1(t)$ takes on the second 1-value, but $d_2(t)$ is still 1, so $\theta_1(t) = \pi/4$. At $3T_b$ seconds, $d_2(t)$ changes to -1, so $\theta_2(t) = 7\pi/4$. At $4T_b$ seconds, $d_1(t)$ changes to -1, so $\theta_2(t) = 7\pi/4$, etc.

Problem 8.3
For QPSK,

$$P_{E,\text{ symbol}} = 2Q\left(\sqrt{E_s/N_0}\right) = 10^{-5}$$

Trial and error using the asymptotic approximation for the Q-function gives $[E_s/N_0]_{\text{req'd}} \approx$ 12.9 dB = 19.5. If the quadrature-carrier amplitudes are A, then the amplitude of the envelope-phase-modulated form of the carrier is $\sqrt{2}A$, and $E_s/N_0 = \left(\sqrt{2}A\right)^2 T/(2N_0) = A^2/N_0R$. Hence, $A_{\text{req'd}} = \sqrt{N_0R\,[E_s/N_0]_{\text{req'd}}} = \sqrt{(10^{-11})(19.5)R} = 1.4\times 10^{-5}\sqrt{R}$. The answers are as follows: (a) 0.0014 V; (b) 0.0031 V; (c) 0.0044 V; (d) 0.0099 V; (e) 0.014 V; (f) 0.0442 V.

Problem 8.4
Take the expectation of the product after noting that the average values are 0 because $\mathrm{E}[n(t)] = 0$:

$$\begin{aligned}
E[N_1N_2] &= E\left[\int_0^{T_s} n(t)\cos(\omega_c t)\,dt \int_0^{T_s} n(t)\sin(\omega_c t)\,dt\right] \\
&= \int_0^{T_s}\int_0^{T_s} E[n(t)\,n(\lambda)]\cos(\omega_c t)\sin(\omega_c\lambda)\,dt d\lambda \\
&= \int_0^{T_s}\int_0^{T_s} \frac{N_0}{2}\delta(t-\lambda)\cos(\omega_c t)\sin(\omega_c\lambda)\,dt d\lambda \\
&= \frac{N_0}{2}\int_0^{T_s}\cos(\omega_c t)\sin(\omega_c t)\,dt \\
&= \frac{N_0}{4}\int_0^{T_s}\sin(2\omega_c t)\,dt = 0
\end{aligned}$$

Problem 8.5
(a) Use

$$\theta_i(t) = \tan^{-1}\left[\frac{d_2(t)}{d_1(t)}\right]$$

(i) If $\theta_i(t) = 45^{\circ}$, $d_1(t) = 1$ and $d_2(t) = 1$; (ii) If $\theta_i(t) = 135^{\circ}$, $d_1(t) = -1$ and $d_2(t) = 1$; (iii) If $\theta_i(t) = -45^{\circ}$, $d_1(t) = 1$ and $d_2(t) = -1$; (iv) If $\theta_i(t) = -135^{\circ}$; $d_1(t) = -1$ and $d_2(t) = -1$.
(b) Error in detecting $d_1(t)$: (i) $\theta_i(t) = 135^{\circ}$; (ii) $\theta_i(t) = 45^{\circ}$; (iii) $\theta_i(t) = -135^{\circ}$; (iv) $\theta_i(t) = -45^{\circ}$.

(c) Error in detecting $d_2(t)$; (i) $\theta_i(t) = -45^o$; (ii) $\theta_i(t) = -135^o$; (iii) $\theta_i(t) = 45^o$; (iv) $\theta_i(t) = 135^o$.

Problem 8.6
(a) Both are equivalent in terms of symbol error probabilities. (b) QPSK is 3 dB worse in terms of symbol error probability for equal transmission bandwidths, but it handles twice as many bits per second. (c) Choose QPSK over BPSK in terms of performance; however, other factors might favor BPSK, such as simpler implementation.

Problem 8.7
The exact result is

$$\begin{aligned} P_{\text{symbol}} &= 1-(1-P_{E_1})^2 \\ &= 2P_{E_1} - P_{E_1}^2 \end{aligned}$$

The approximation is $P_{\text{symbol}} \approx 2P_{E_1}$, so the error term is

$$\epsilon = P_{E_1}^2 = \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

Clearly, since the Q-function is monotonically decreasing with increasing E_s/N_0, the error term becomes negligible compared with $P_{E_1} = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ as E_s/N_0 becomes larger.

Problem 8.8
For the data stream 11100 10111 00100 00011, the quadrature data streams are

$$\begin{aligned} d_1(t) &= \text{11-1-11-11-1-11} \\ d_2(t) &= \text{1-1111-1-1-1-11} \end{aligned}$$

Each 1 above means a positive rectangular pulse T_s seconds in duration and each -1 above means a negative rectangular pulse T_s seconds in duration. For QPSK, these pulse sequences are aligned and for OQPSK, the one corresponding to $d_2(t)$ is delayed by $T_s/2 = T_b$ seconds with respect to $d_1(t)$. Type I MSK corresponds to the OQPSK waveforms $d_1(t)$ and $d_2(t)$ multiplied by cosine and sine waveforms with periods of $2T_s$, respectively, and type II MSK corresponds to $d_1(t)$ and $d_2(t)$ multiplied by absolute-value cosine and sine waveforms with periods of T_s, respectively (one half cosine or sine per T_s pulse).
Waveforms for QPSK, OQPSK, and Type II MSK generated by a MATLAB program are shown in Figures 8.1 - 8.3 for a random (coin toss) serial bit sequence.

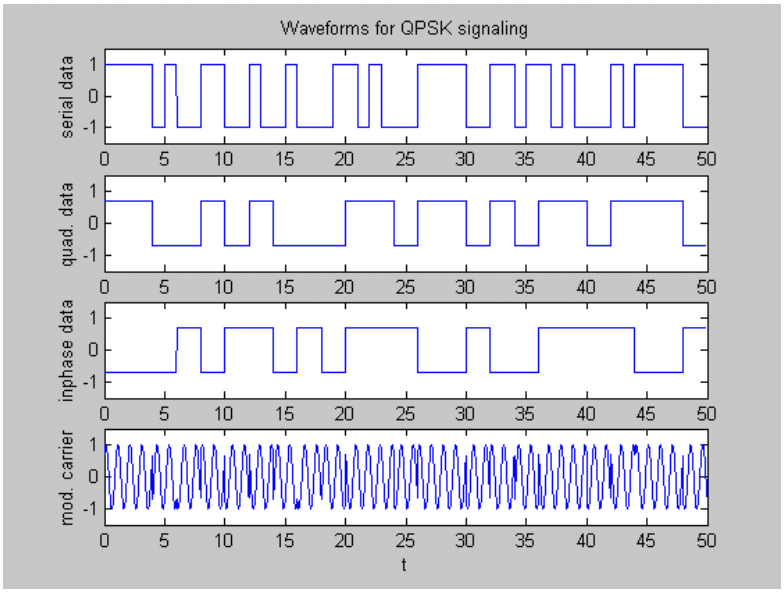

Figure 8.1:

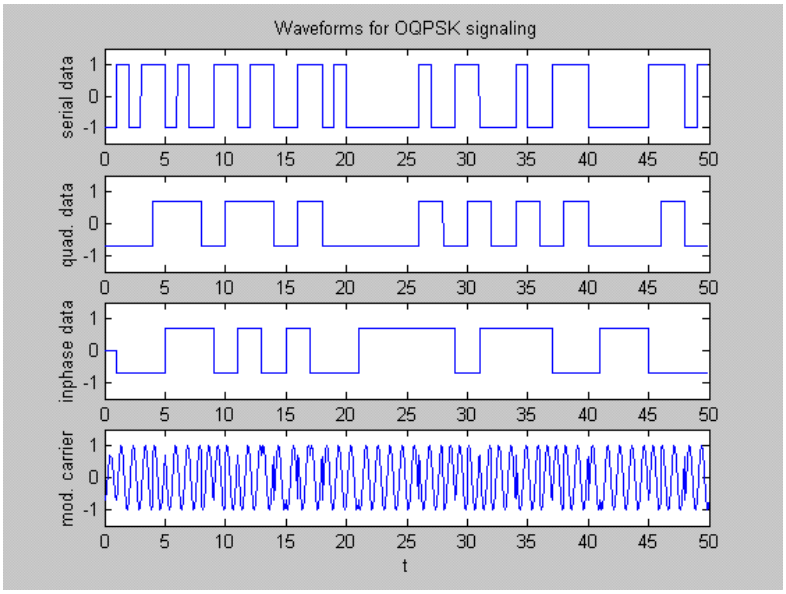

Figure 8.2:

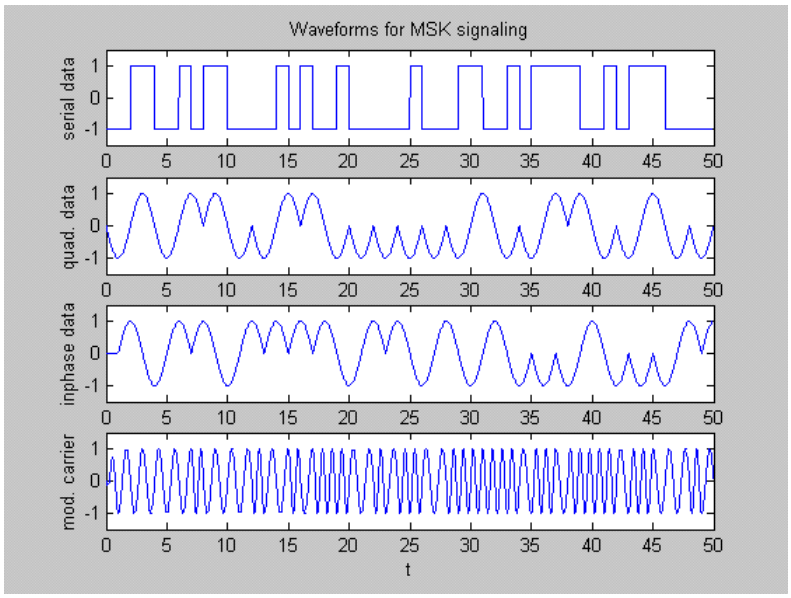

Figure 8.3:

Problem 8.9
For QPSK, the excess phase corresponds to a stepwise function that may change values each T_s seconds. The phase deviation is given by

$$\theta_i(t) = \tan^{-1}\left[\frac{d_2(t)}{d_1(t)}\right]$$

so the sequence of phases for QPSK computed from $d_1(t)$ and $d_2(t)$ given in Problem 8.8 is $\pi/4$, $-\pi/4$, $\pi/4$, $3\pi/4$, $\pi/4$, $-3\pi/4$, $-\pi/4$, $-3\pi/4$, $-3\pi/4$, $\pi/4$ radians. For OQPSK, the phase can change each $T_s/2 = T_b$ seconds because $d_2(t)$ is delayed by $T_s/2$ seconds with respect to $d_1(t)$. The maximum phase change is $\pm\pi/2$ radians. For MSK, the excess phase trajectories are straight lines of slopes $\pm\pi/2T_b$ radians/second.

Problem 8.10
Write the sinc-functions as $\sin(x)/x$ functions. Use appropriate trigonometric identities to reduce the product of the sinc-functions to the given form.

Problem 8.11
If $d(t)$ is a sequence of alternating 1s and 0s, the instantaneous frequency is $1/4T_b$ Hz above the carrier (with the definition of $\theta_i(t)$ given by (8.17) and (8.18)). If it is a sequence of

1s or 0s, the instantaneous frequency is $1/4T_b$ Hz below the carrier. (a) The instantaneous frequency is $5,000,000 + 100,000/4 = 1,025,000$ Hz. (b) The instantaneous frequency is $5,000,000 - 100,000/4 = 975,000$ Hz.

Problem 8.12

The signal points lie on a circle of radius $\sqrt{E_s}$ centered at the origin equally spaced at angular intervals of 22.5 degrees (360/16). The decision regions are pie wedges centered over each signal point.

Problem 8.13

The bounds on symbol error probability are given by

$$P_1 \leq P_{E,\text{ symbol}} \leq 2P_1$$

where

$$P_1 = Q\left[\sqrt{\frac{2E_s}{N_0}}\sin\left(\pi/M\right)\right]$$

For moderate signal-to-noise ratios and $M \geq 8$, the actual error probability is very close to the upper bound. Assuming Gray encoding, the bit error probability is given in terms of symbol error probability as

$$P_{E,\text{ bit}} \approx \frac{P_{E,\text{ symbol}}}{\log_2\left(M\right)}$$

and the energy per bit-to-noise spectral density ratio is

$$\frac{E_b}{N_0} = \frac{1}{\log_2\left(M\right)}\frac{E_s}{N_0}$$

Thus, we solve

$$\frac{2}{\log_2\left(M\right)}Q\left[\sqrt{2\log_2\left(M\right)\frac{E_b}{N_0}}\sin\left(\pi/M\right)\right] = 10^{-5}$$

or

$$Q\left[\sqrt{2\log_2\left(M\right)\frac{E_b}{N_0}}\sin\left(\pi/M\right)\right] = \frac{10^{-5}}{2}\times\log_2\left(M\right) = \begin{cases} 1.5\times10^{-5},\ M=8 \\ 2\times10^{-5},\ M=16 \\ 2.5\times10^{-5},\ M=32 \\ 3\times10^{-5},\ M=64 \end{cases}$$

The argument of the Q-function to give these various probabilities are found, approximately, by trial and error (either using MATLAB and a program for the Q-function or a calculator and the asymptotic expansion of the Q-function):

$$\text{Q-function argument} = \begin{cases} 4.17, \ M = 8 \\ 4.1, \ M = 16 \\ 4.06, \ M = 32 \\ 4.02, \ M = 64 \end{cases}$$

Thus, for $M = 8$, we have

$$\begin{aligned} \sqrt{2\log_2(8)\frac{E_b}{N_0}}\sin(\pi/8) &= 4.17 \\ \sqrt{6\frac{E_b}{N_0}} \times 0.3827 &= 4.17 \\ \frac{E_b}{N_0} &= 10\log_{10}\left[\frac{1}{6}\left(\frac{4.17}{0.3827}\right)^2\right] = 12.96 \text{ dB} \end{aligned}$$

For $M = 16$, we have

$$\begin{aligned} \sqrt{2\log_2(16)\frac{E_b}{N_0}}\sin(\pi/16) &= 4.1 \\ \sqrt{8\frac{E_b}{N_0}} \times 0.1951 &= 4.17 \\ \frac{E_b}{N_0} &= 10\log_{10}\left[\frac{1}{8}\left(\frac{4.17}{0.1951}\right)^2\right] = 17.57 \text{ dB} \end{aligned}$$

For $M = 32$, we have

$$\begin{aligned} \sqrt{2\log_2(32)\frac{E_b}{N_0}}\sin(\pi/32) &= 4.06 \\ \sqrt{10\frac{E_b}{N_0}} \times 0.098 &= 4.06 \\ \frac{E_b}{N_0} &= 10\log_{10}\left[\frac{1}{10}\left(\frac{4.06}{0.098}\right)^2\right] = 22.35 \text{ dB} \end{aligned}$$

For $M = 64$, we have

$$\begin{aligned}
\sqrt{2\log_2(64)\frac{E_b}{N_0}}\sin(\pi/64) &= 4.02 \\
\sqrt{12\frac{E_b}{N_0}} \times 0.049 &= 4.02 \\
\frac{E_b}{N_0} &= 10\log_{10}\left[\frac{1}{12}\left(\frac{4.02}{0.049}\right)^2\right] = 27.49 \text{ dB}
\end{aligned}$$

Problem 8.14

The binary number representation for the decimal digits $0-15$ and their Gray code equivalents are given below:

Bin. No.	Gray	Bin. No.	Gray	Bin. No.	Gray	Bin. No.	Gray
0000	0000	0100	0110	1000	1100	1100	1010
0001	0001	0101	0111	1001	1101	1101	1011
0010	0011	0110	0101	1010	1111	1110	1001
0011	0010	0111	0100	1011	1110	1111	1000

Problem 8.15

Let the coordinates of the received data vector, given signal labeled 1111 was sent, be

$$(X,\ Y) = (a + N_1,\ a + N_2)$$

where N_1 and N_2 are zero-mean, uncorrelated Gaussian random variables with variances $N_0/2$. Since they are uncorrelated, they are also independent. Thus,

$$\begin{aligned}
P(C \mid I) &= P(0 \le X \le 2a)\, P(0 \le Y \le 2a) \\
&= \int_0^{2a} \frac{e^{-(x-a)^2/N_0}}{\sqrt{\pi N_0}} dx \int_0^{2a} \frac{e^{-(y-a)^2/N_0}}{\sqrt{\pi N_0}} dy
\end{aligned}$$

Let

$$u = \frac{\sqrt{2}\,(x-a)}{N_0} \text{ and } u = \frac{\sqrt{2}\,(y-a)}{N_0}$$

This results in

$$\begin{aligned}
P(C \mid \mathrm{I}) &= \int_{-\sqrt{\frac{2a^2}{N_0}}}^{\sqrt{\frac{2a^2}{N_0}}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \int_{-\sqrt{\frac{2a^2}{N_0}}}^{\sqrt{\frac{2a^2}{N_0}}} \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv \\
&= 4 \int_{0}^{\sqrt{\frac{2a^2}{N_0}}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \int_{0}^{\sqrt{\frac{2a^2}{N_0}}} \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv \\
&= \left[1 - 2Q\left(\sqrt{\frac{2a^2}{N_0}}\right)\right]^2
\end{aligned}$$

Similar derivations can be carried out for the type II and III regions.

Problem 8.16

The symbol error probability expression is

$$P_s = 1 - \left[\frac{1}{4}P(C \mid \mathrm{I}) + \frac{1}{2}P(C \mid \mathrm{II}) + \frac{1}{4}P(C \mid \mathrm{III})\right]$$

where

$$\begin{aligned}
P(C \mid \mathrm{I}) &= [1 - 2Q(A)]^2, \; A = \sqrt{\frac{2a^2}{N_0}} \\
P(C \mid \mathrm{II}) &= [1 - 2Q(A)][1 - Q(A)] \\
P(C \mid \mathrm{III}) &= [1 - Q(A)]^2
\end{aligned}$$

Thus

$$\begin{aligned}
P_E &= 1 - \left\{\frac{1}{4}[1 - 2Q(A)]^2 + \frac{1}{2}[1 - 2Q(A)][1 - Q(A)] + \frac{1}{4}[1 - Q(A)]^2\right\} \\
&= 1 - \left\{\frac{1}{4}\left[1 - 4Q(A) + 4Q^2(A)\right] + \frac{1}{2}\left[1 - 3Q(A) + 2Q^2(A)\right] + \frac{1}{4}\left[1 - 2Q(A) + Q^2(A)\right]\right\} \\
&\approx 1 - \{1 + 3Q(A)\}, \text{ neglecting all terms in } Q^2(A) \\
&= 3Q\left(\sqrt{\frac{2a^2}{N_0}}\right)
\end{aligned}$$

Problem 8.17
This follows by counting type I, II, and III regions. There are $\sqrt{M} \times \sqrt{M}$ total regions, four of which are type III regions (the corner regions), $4\left(\sqrt{M}-2\right)$ type II regions, and $\left(\sqrt{M}-2\right)^2$ type I regions. A sketch will help in determining this. Thus, the given expression for P_E follows. The expression for a follows by computing the average symbol energy in terms of a. To accomplish this, the sums

$$\sum_{i=1}^{m} i = \frac{m\,(m+1)}{2} \text{ and } \sum_{i=1}^{m} i^2 = \frac{m\,(m+1)\,(2m+1)}{6}$$

are convenient. The approximate probability of error expression follows in a manner similar to the derivation for $M = 16$ outlined in Problem 8.16.

Problem 8.18
Use

$$P_{E,\text{ bit}} = \frac{M}{2\,(M-1)} P_{E,\text{ symbol}} \text{ and } \frac{E_b}{N_0} = \frac{1}{\log_2(M)} \frac{E_s}{N_0}$$

A tight bound for the symbol error probability for coherent M-ary FSK is

$$P_{E,\text{ symbol}} \leq MQ\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Use the asymptotic expression for the Q-function and iteration on a calculator to get the following results:

For $M = 8$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 7.5$ dB;
For $M = 16$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 6.5$ dB;
For $M = 32$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 5.9$ dB;
For $M = 64$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 5.3$ dB.

Problem 8.19
Use the same relations as in Problem 8.18 for relating signal-to-noise ratio per bit and symbol and for relating bit to symbol error probability, except now

$$P_{E,\text{ symbol}} = \sum_{k=1}^{M-1} \binom{M-1}{k} \frac{(-1)^{k+1}}{k+1} \exp\left(-\frac{k}{k+1}\frac{E_s}{N_0}\right)$$

Use MATLAB to perform the sum. The following results are obtained:

For $M = 2$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 12.3$ dB;

For $M = 4$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 9.6$ dB;
For $M = 8$, $P_{E,\text{ bit}} = 10^{-4}$ for $E_b/N_0 = 8.2$ dB.

Problem 8.20
This is a normal cartesian coordinate system with the signal points located on the ϕ_1 and ϕ_2 axes at $\sqrt{E_s}$. The decision regions are formed by the positive coordinate axes and the 45-degree bisector of the first quadrant angle.

Problem 8.21
The bandwidth efficiencies are given by

$$\frac{R}{B} = \begin{cases} 0.5\log_2(M), & M\text{-PSK and } M\text{-QAM} \\ \frac{\log_2(M)}{M+1}, & \text{coherent } M\text{-FSK} \end{cases}$$

This gives the results given in the table below:

$$\text{PSK: } \frac{R}{B} = \begin{cases} 1.5 \text{ bps/Hz}, M = 8 \\ 2.0 \text{ bps/Hz}, M = 16 \\ 2.5 \text{ bps/Hz}, M = 32 \end{cases}; \qquad B = \begin{cases} 6.67\ k\text{Hz}, M = 8 \\ 5.0 \text{ kHz}, M = 16 \\ 4.0 \text{ kHz}, M = 32 \end{cases}$$

$$\text{16-QAM: } \frac{R}{B} = 2.0 \text{ bps/Hz}, M = 16; \qquad B = 5.0 \text{ kHz}$$

$$\text{FSK: } \frac{R}{B} = \begin{cases} 0.33 \text{ bps/Hz}, M = 8 \\ 0.24 \text{ bps/Hz}, M = 16 \\ 0.15 \text{ bps/Hz}, M = 32 \end{cases}; \qquad B = \begin{cases} 30.0 \text{ kHz}, M = 8 \\ 42.6 \text{ kHz}, M = 16 \\ 66.0 \text{ kHz}, M = 32 \end{cases}$$

Problem 8.22
Use Figure 8.19 noting that the abscissa is normalized baseband bandwidth. RF bandwidthis twice as large. The 90% power containment bandwidth is defined by the ordinate = -10 dB. (c) $B_{90,\text{ BPSK}} = 1.6/T_b = 1.6R_b$; (b) $B_{90,\text{ QPSK, OQPSK}} = 0.8/T_b = 0.8R_b$; (a) $B_{90,\text{ MSK}} = 0.8/T_b = 0.8R_b$.

Problem 8.23
Use Figure 8.19 noting that the abscissa is normalized baseband bandwidth. RF bandwidth is twice as great. The 99% power containment bandwidth is defined by the ordinate = -20 dB. (a) $B_{99,\text{ BPSK}} = 14/T_b = 14R_b$; (b) $B_{99,\text{ QPSK, OQPSK}} = 3.5/T_b = 3.5R_b$; (c) $B_{99,\text{ MSK}} = 1.23/T_b = 1.23R_b$.

Problem 8.24
The baseband power spectrum is

$$G(f) = 2A^2T_b\left[\log_2(M)\right]\text{sinc}^2\left[\log_2(M)\,T_bf\right]$$

where

$$\begin{aligned} A^2 &= E_s\overline{\cos^2\theta_i} = E_s\overline{\sin^2\theta_i} \\ &= \frac{1}{M}\sum_{i=1}^{M}\cos^2\left[\frac{2\pi\left(i-1\right)}{M}\right] = \frac{1}{M}\sum_{i=1}^{M}\sin^2\left[\frac{2\pi\left(i-1\right)}{M}\right] \end{aligned}$$

The 10% power containment bandwidth is defined by (see Problem 8.22)

$$\begin{aligned} \log_2\left(M\right)B_{90,\text{ MPSK}} &= 0.8/T_b = 0.8R_b \\ B_{90,\text{ MPSK}} &= 0.8R_b/\log_2\left(M\right) = \begin{cases} 0.267R_b \text{ Hz},\ M=8 \\ 0.25R_b \text{ Hz},\ M=16 \\ 0.16R_b \text{ Hz},\ M=8 \end{cases} \end{aligned}$$

Problem 8.25

(a) For NRZ mark:

$$R_m = \begin{cases} \left(A\right)\left(A\right)\frac{1}{2} + \left(-A\right)\left(-A\right)\frac{1}{2} = A^2,\ m=0 \\ \left(A\right)\left(A\right)\frac{1}{4} + \left(-A\right)\left(A\right)\frac{1}{4} + \left(A\right)\left(-A\right)\frac{1}{4} + \left(-A\right)\left(-A\right)\frac{1}{4} = 0,\ m\neq 0 \end{cases}$$

$$S_r\left(f\right) = T_b\,\text{sinc}^2\left(fT_b\right)$$

$$S_{\text{NRZ, mark}}\left(f\right) = A^2T_b\,\text{sinc}^2\left(fT_b\right)$$

(b) For polar RZ:

$$R_m = \begin{cases} A^2,\ m=0 \\ 0,\ m\neq 0 \end{cases}$$

$$S_r\left(f\right) = \frac{T_b}{2}\,\text{sinc}^2\left(\frac{fT_b}{2}\right)$$

$$S_{\text{polar RZ}}\left(f\right) = \frac{A^2T_b}{4}\,\text{sinc}^2\left(\frac{fT_b}{2}\right)$$

(c) Bipolar RZ:

$$R_m = \begin{cases} (A)(A)\frac{1}{2} + (0)(0)\frac{1}{2} = \frac{A^2}{2},\ m = 0 \\ (A)(A)\left(\frac{1}{4}\right)^2 + (A)(-A)\left(\frac{1}{4}\right)^2 + (-A)(A)\left(\frac{1}{4}\right)^2 + (-A)(-A)\left(\frac{1}{4}\right)^2 + (A)(0)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \\ + (0)(A)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (-A)(0)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + (0)(-A)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (0)(0)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 0,\ m \neq 0 \end{cases}$$

$$S_r(f) = \frac{T_b}{2}\operatorname{sinc}^2\left(\frac{fT_b}{2}\right)$$

$$S_{\text{bipolar RZ}}(f) = \frac{A^2T_b}{8}\operatorname{sinc}^2\left(\frac{fT_b}{2}\right)$$

Problem 8.26

To show:

$$\cos\left(\frac{\pi t}{2T_b}\right)\Pi\left(\frac{t}{2T_b}\right) \longleftrightarrow \frac{4T_b}{\pi}\frac{\cos(2\pi T_b f)}{1-(4T_b f)^2}$$

Use the modulation theorem together with the Fourier transform of a rectangular pulse to get

$$\cos\left(\frac{\pi t}{2T_b}\right)\Pi\left(\frac{t}{2T_b}\right) \longleftrightarrow T_b\operatorname{sinc}[2T_b(f-1/4T_b)] + T_b\operatorname{sinc}[2T_b(f+1/4T_b)]$$

Rewrite the RHS as

$$\text{RHS} = T_b\left[\frac{\sin(2\pi T_b f - \pi/2)}{2\pi T_b f - \pi/2} + \frac{\sin(2\pi T_b f + \pi/2)}{2\pi T_b f + \pi/2}\right]$$

Note that

$$\sin(2\pi T_b f \pm \pi/2) = \pm\cos(2\pi T_b f)$$

Then

$$\begin{aligned} \text{RHS} &= \frac{2T_b}{\pi}\left[\frac{1}{1-4T_b f} + \frac{1}{1+4T_b f}\right]\cos(2\pi T_b f) \\ &= \frac{4T_b}{\pi}\frac{\cos(2\pi T_b f)}{1-(4T_b f)^2} \end{aligned}$$

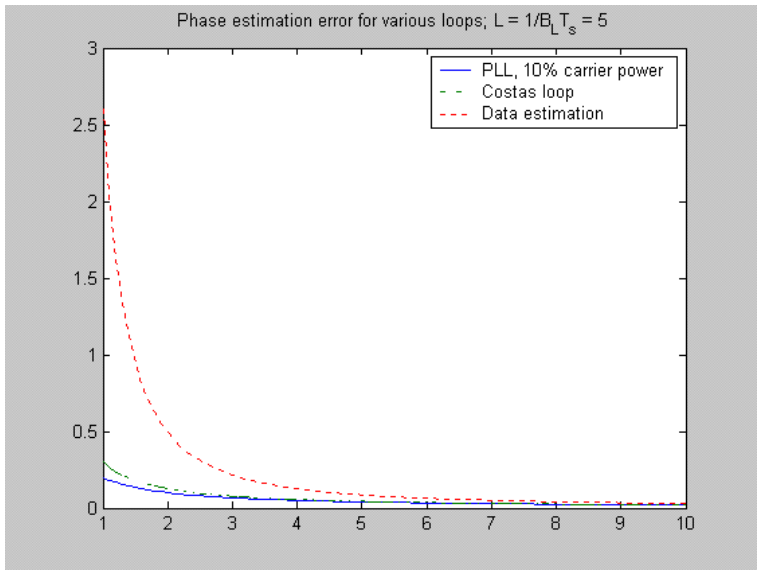

Figure 8.4:

Problem 8.27
Apply (5.28).

Problem 8.28
This is a matter of MATLAB programming and plotting.

Problem 8.29
A plot is given in Figure 8.4.

Problem 8.30
The states are (read from top to bottom, column by column):

1111	0100	1011
0111	0010	0101
0011	1001	1010
0001	1100	1101
1000	0110	1110

The output sequence is the last digit in each 4-element word. The autocorrelation function is a triangular pulse centered at the origin two units wide repeated every 15 units and a horzontal line between the triangles at $-1/15$ units.

Problem 8.31

The states are (read from top to bottom, column by column):

11111	00101	10100	01101
01111	10010	01010	00110
00111	01001	11010	00011
10011	00100	11101	10001
11001	00010	01110	11000
01100	00001	01110	11100
10110	10000	10111	11110
01011	01000	11011	

The output sequence is the last digit in each 5-element word. The autocorrelation function is a triangular pulse centered at the origin two units wide repeated every 31 units and a horzontal line between the triangles at $-1/31$ units.

Problem 8.32

To find the aperiodic correlation function, we slide the sequence past itself and compute the number of alike bits minus the number of unalike bits for the overlap (the sequence is not periodically repeated). The result can be scaled by the length of the sequence if desired. (a) -1, 0, -1, 0, -1, 0, 7, 0, -1, 0, -1, 0, -1 (the maximum absolute value correlation is 1 - nonzero delay); (b) -1, 0, -1, 0, 3, 0, 1, -2, -1, -4, -1, 0, -1, 0, 15, . . . (the maximum absolute value correlation is 4 - nonzero delay).

Problem 8.33

Note that the expectation of N_g is zero because $n(t)$ has zero mean. Write the expectation of the square of N_g as an iterated integral. The expectation can be taken inside. Use the fact that

$$E\left[n(t)\,n(\lambda)\right] = \frac{N_0}{2}\delta(t-\lambda)$$

to reduce the double integral to a single integral. The integral of the resulting cosine squared is $T_b/2$. The result for the variance (same as the mean square) is then found to be $N_0 T_b$.

Problem 8.34

The random variable to be considered is

$$N_I = \int_0^{T_b} A_I c(t) \cos(\Delta\omega t + \theta - \phi)\, dt$$

where $\theta - \phi$ is assumed uniform in$[0,\ 2\pi)$. Clearly, the expectation of N_I is zero due to this random phase. The variance is the same as the mean square, which is given by

$$\begin{aligned}
\operatorname{var}(N_I) &= E\left\{\int_0^{T_b}\int_0^{T_b} A_I^2 c(t)\, c(\lambda) \cos(\Delta\omega t + \theta - \phi) \cos(\Delta\omega\lambda + \theta - \phi)\, dt d\lambda\right\} \\
&= \int_0^{T_b}\int_0^{T_b} A_I^2 E\left[c(t)\, c(\lambda)\right] \cos(\Delta\omega t + \theta - \phi) \cos(\Delta\omega\lambda + \theta - \phi)\, dt d\lambda \\
&= A_I^2 T_c \int_0^{T_b}\int_0^{T_b} \frac{1}{T_c}\Lambda\left[(t-\lambda)/T_c\right] \frac{1}{2}\cos\left[\Delta\omega(t-\lambda)\right] dt d\lambda \\
&= A_I^2 T_c \int_0^{T_b} \frac{1}{2} dt = \frac{A_I^2 T_c T_b}{2}
\end{aligned}$$

Problem 8.35

(a) The processing gain is

$$G_p = \frac{T_b}{T_c} = \frac{R_c}{R_b} \quad \text{or} \quad R_c = G_p R_b$$

For $G_p = 100$ and a data rate of 1 Mbps, $R_c = 100$ megachips per second; (b) $B_{RF} = 2R_c =$ 200 MHz; (c) Use (8.99) to get the following:

$P_E = 5.03 \times 10^{-5}$ for a JSR of 5 dB;
$P_E = 8.34 \times 10^{-4}$ for a JSR of 10 dB;
$P_E = 1.42 \times 10^{-2}$ for a JSR of 15 dB;
$P_E = 0.34$ for a JSR of 30 dB.

Problem 8.36

The results are

$P_E = 5.72 \times 10^{-6}$ for a JSR of 5 dB;
$P_E = 1.09 \times 10^{-5}$ for a JSR of 10 dB;
$P_E = 5.03 \times 10^{-5}$ for a JSR of 15 dB;
$P_E = 0.0895$ for a JSR of 30 dB.

Problem 8.37

In the limit at SNR $E_b/N_0 \to \infty$, the argument of the Q-function for $P_E = Q\left(\sqrt{\text{SNR}}\right)$ becomes

$$\sqrt{\text{SNR}} = \sqrt{\frac{3N}{K-1}} \approx 3.81 \text{ to give } P_E = 10^{-4}$$

Thus,

$$\begin{aligned} K &= \frac{3N}{(3.81)^2} + 1 \\ &= 53.7 \end{aligned}$$

Round this down to $K = 53$ users since the number of users must be an integer.

Problem 8.38
The result is

$$T_{\text{acq}} = \frac{nT_cT_e}{2\left(1 - P_{\text{FA}}\right)} \frac{2 - P_{\text{H}}}{P_{\text{H}}} = \frac{2\left(1000\right)\left(10^{-3}\right)}{2\left(1 - 10^{-3}\right)} \frac{2 - 0.9}{0.9} = 1.22 \text{ seconds}$$

Problem 8.39
(a) Only the OBP case will be done. See the text for the bent-pipe case. Equation (8.121) becomes

$$p_d = \frac{10^{-5} - p_u}{1 - 2p_u}$$

For BPSK, the uplink and downlink probabilities of error are related to the uplink and downlink SNRs by

$$\begin{aligned} p_u &= Q\left[\sqrt{2\left(\frac{E_b}{N_0}\right)_u}\right] = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\left(\frac{E_b}{N_0}\right)_u}\right)\right] \\ p_d &= Q\left[\sqrt{2\left(\frac{E_b}{N_0}\right)_d}\right] = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\left(\frac{E_b}{N_0}\right)_d}\right)\right] \end{aligned}$$

where these probabilities have been put in terms of the error function because MATLAB includes an inverse error function. When these relations are inverted, these become

$$\begin{aligned} \left(\frac{E_b}{N_0}\right)_u &= \left[\text{erf}^{-1}\left(1 - 2p_u\right)\right]^2 \\ \left(\frac{E_b}{N_0}\right)_d &= \left[\text{erf}^{-1}\left(1 - 2p_d\right)\right]^2 \end{aligned}$$

Typical values are given in the table below. The MATLAB program for computing them is also given below along with a plot. Part (b) is just a matter of inputting the desired p_overall in the program.

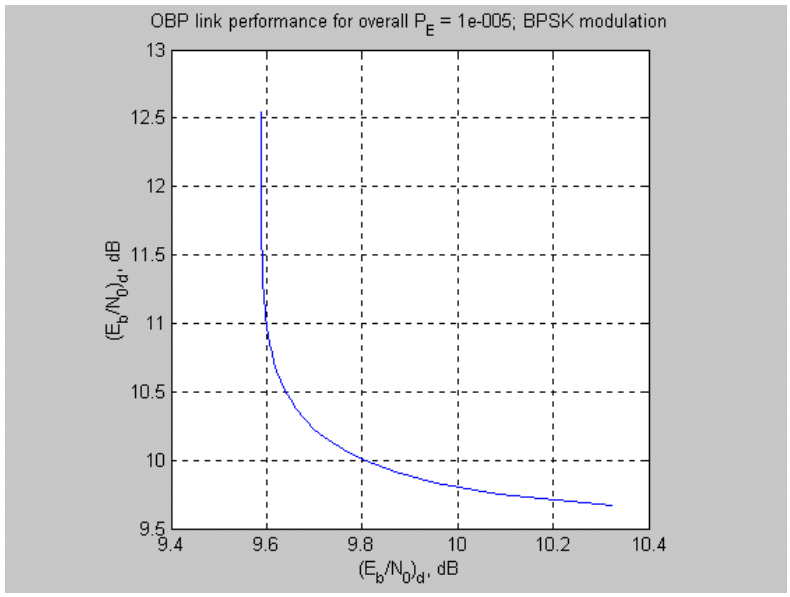

Figure 8.5:

p_u	$(E_b/N_0)_u$, dB	p_d	$(E_b/N_0)_d$, dB
10^{-9}	12.5495	9.999×10^{-6}	9.5879
3×10^{-9}	12.3228	9.997×10^{-6}	9.5880
7×10^{-9}	12.0839	9.993×10^{-6}	9.5882
4.3×10^{-8}	11.5639	9.957×10^{-6}	9.5898
7.2×10^{-7}	10.6497	9.280×10^{-6}	9.6217

```
% Solution for Problem 8.39
%
p_overall = input('Enter desired end-to-end probability of error ')
pu = logspace(log10(p_overall)-4, log10(p_overall)+1e-20);
pd = (p_overall - pu)./(1 - 2*pu);
Eb_N0_u = 20*log10(erfinv(1 - 2*pu));
Eb_N0_d = 20*log10(erfinv(1 - 2*pd));
plot(Eb_N0_d, Eb_N0_u), xlabel('(E_b/N_0)_d, dB'), ylabel('(E_b/N_0)_d, dB'),...
grid, axis square
```

Problem 8.40

(a) This is similar to the previous problem except that the error probability expresseion for noncoherent FSK is

$$P_E = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

Solving for $\frac{E_b}{N_0}$, we have

$$\frac{E_b}{N_0} = -2\ln(2P_E)$$

Thus,

$$\begin{aligned} p_d &= \frac{10^{-6} - p_u}{1 - 2p_u} \\ \left(\frac{E_b}{N_0}\right)_u &= -2\ln(2p_u) \\ \left(\frac{E_b}{N_0}\right)_d &= -2\ln(2p_d) \end{aligned}$$

(b) For DPSK, the probability of error is

$$P_E = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

Therefore, the equations describing the link are

$$\begin{aligned} p_d &= \frac{10^{-6} - p_u}{1 - 2p_u} \\ \left(\frac{E_b}{N_0}\right)_u &= -\ln(2p_u) \\ \left(\frac{E_b}{N_0}\right)_d &= -\ln(2p_d) \end{aligned}$$

A MATLAB program for computing the performance curves for both parts (a) and (b) is given below. Typical performance curves are also shown.

```
% Solution for Problem 8.40
%
mod_type = input('Enter 1 for NFSK; 2 for DPSK ')
p_overall = input('Enter desired end-to-end probability of error ')
pu = logspace(log10(p_overall)-6, log10(1.000001*p_overall));
```

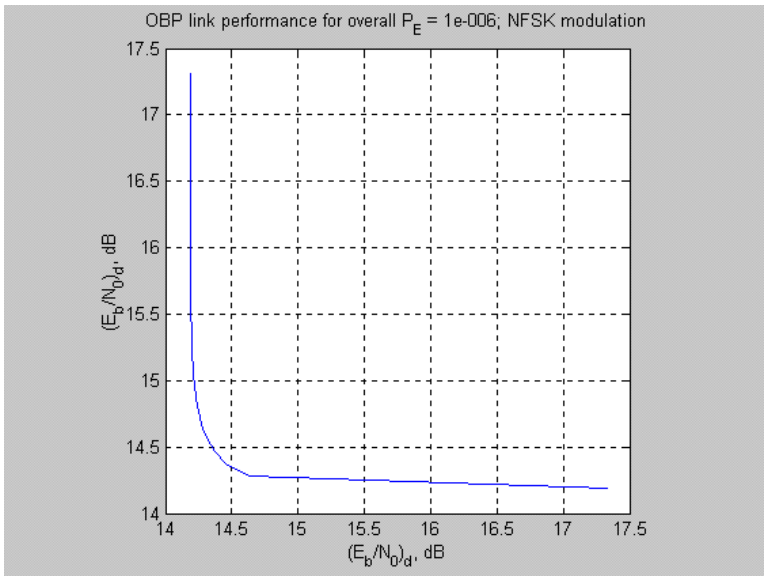

Figure 8.6:

```
pd = (p_overall - pu)./(1 - 2*pu);
if mod_type == 1
    Eb_N0_u = 10*log10(-2*log(2*pu));
    Eb_N0_d = 10*log10(-2*log(2*pd));
elseif mod_type == 2
    Eb_N0_u = 10*log10(-log(2*pu));
    Eb_N0_d = 10*log10(-log(2*pd));
end
plot(Eb_N0_d, Eb_N0_u), xlabel('(E_b/N_0)_d, dB'), ylabel('(E_b/N_0)_d, dB'),...
    grid, axis square
if mod_type == 1
    title(['OBP link performance for overall P_E = ', num2str(p_overall), '; NFSK
modulation'])
elseif mod_type == 2
    title(['OBP link performance for overall P_E = ', num2str(p_overall), '; DPSK
modulation'])
end
```

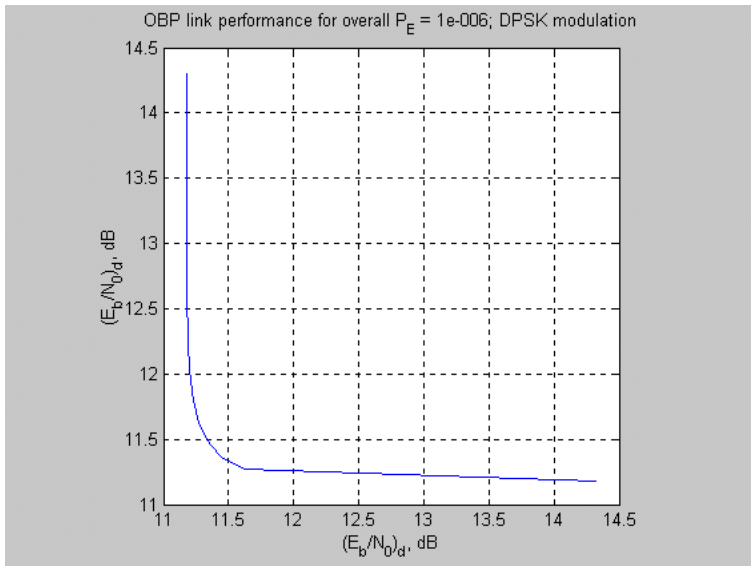

Figure 8.7:

Problem 8.41

For noncoherent FSK, the bit error probability, using (8.51) and (8.54), is

$$P_b = \frac{M^2}{2(M-1)} Q\left(\sqrt{\log_2(M)\frac{E_b}{N_0}}\right) \approx \frac{M}{2} Q\left(\sqrt{\log_2(M)\frac{E_b}{N_0}}\right)$$

So, the equations for the OBP link are

$$\begin{aligned}
p_d &= \frac{10^{-6} - p_u}{1 - 2p_u} \\
p_u &\approx \frac{M}{2} Q\left(\sqrt{\log_2(M)\left(\frac{E_b}{N_0}\right)_u}\right) = \frac{M}{4}\left[1 - \operatorname{erf}\left(\sqrt{\frac{\log_2(M)}{2}\left(\frac{E_b}{N_0}\right)_u}\right)\right] \\
\left(\frac{E_b}{N_0}\right)_u &= \left[\frac{2}{\log_2(M)} \operatorname{erf}^{-1}\left(1 - \frac{4p_u}{M}\right)\right]^2 \\
p_d &\approx \frac{M}{2} Q\left(\sqrt{\log_2(M)\left(\frac{E_b}{N_0}\right)_d}\right) = \frac{M}{4}\left[1 - \operatorname{erf}\left(\sqrt{\frac{\log_2(M)}{2}\left(\frac{E_b}{N_0}\right)_d}\right)\right] \\
\left(\frac{E_b}{N_0}\right)_d &= \left[\frac{2}{\log_2(M)} \operatorname{erf}^{-1}\left(1 - \frac{4p_d}{M}\right)\right]^2
\end{aligned}$$

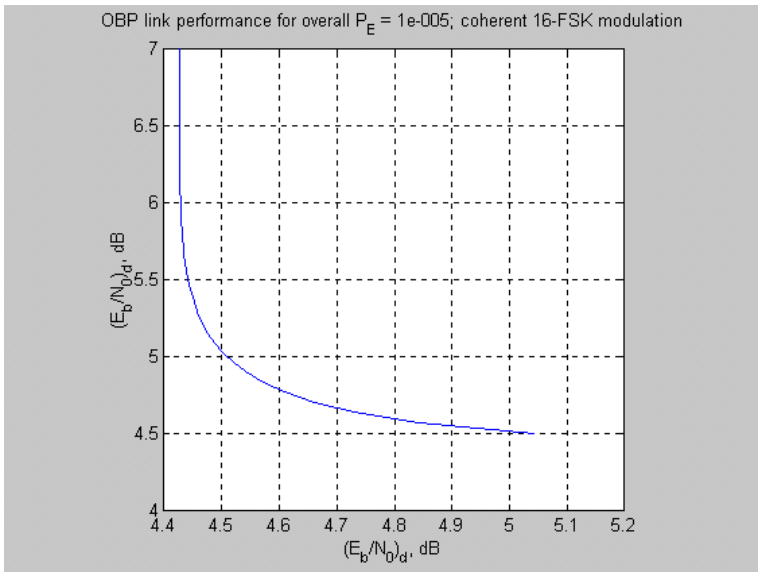

Figure 8.8:

A MATLAB program is given below and a plot for 16-FSK is also given in Fig. 8.8.

```
% Solution for Problem 8.41
%
p_overall = input('Enter desired end-to-end probability of error ')
M = input('Enter the desired M ')
pu = logspace(log10(p_overall)-4, log10(p_overall)+1e-20);
pd = (p_overall - pu)./(1 - 2*pu);
Eb_N0_u = 20*log10((2/log2(M))*erfinv(1 - (4/M)*pu));
Eb_N0_d = 20*log10((2/log2(M))*erfinv(1 - (4/M)*pd));
plot(Eb_N0_d, Eb_N0_u), xlabel('(E_b/N_0)_d, dB'), ylabel('(E_b/N_0)_d, dB'),...
grid, axis square
title(['OBP link performance for overall P_E = ', num2str(p_overall), '; coherent ',
num2str(M),
'-FSK modulation'])
```

Problem 8.42

(a) The number of users per reuse pattern remain the same at 120. For 20 dB minimum signal-to-interference ratio and a propagation power law of $\alpha = 3$, we have

$$20 = 10(3)\log_{10}\left(\frac{D_{\text{co}}}{d_A} - 1\right) - 7.7815$$

or

$$\begin{aligned}
\log_{10}\left(\frac{D_{\text{co}}}{d_A} - 1\right) &= \frac{20 + 7.7815}{30} = 0.794 \\
\frac{D_{\text{co}}}{d_A} &= 10^{0.794} + 1 \\
&= 9.43 = \sqrt{3N}
\end{aligned}$$

or

$$N = \lceil 29.67 \rceil = 31 \quad (i = 1,\ j = 5)$$

The efficiency is

$$\begin{aligned}
\eta_v &= \frac{\left\lfloor \frac{120}{31} \right\rfloor}{6 \text{ MHz}} \\
&= \frac{3}{6} = \frac{1}{2} \text{ voice circuits per base station per MHz}
\end{aligned}$$

(b) For 14 dB minimum signal-to-interference ratio and a propagation power law of $\alpha = 3$, we have

$$\begin{aligned}
14 &= 10(3)\log_{10}\left(\frac{D_{\text{co}}}{d_A} - 1\right) - 7.7815 \\
\log_{10}\left(\frac{D_{\text{co}}}{d_A} - 1\right) &= \frac{14 + 7.7815}{30} = 0.726 \\
\frac{D_{\text{co}}}{d_A} &= 10^{0.726} + 1 \\
&= 6.32 = \sqrt{3N}
\end{aligned}$$

or

$$N = \lceil 13.32 \rceil = 19 \quad (i = 2,\ j = 3)$$

The efficiency is

$$\begin{aligned}
\eta_v &= \frac{\left\lfloor \frac{120}{19} \right\rfloor}{6 \text{ MHz}} \\
&= \frac{6}{6} = 1 \text{ voice circuits per base station per MHz}
\end{aligned}$$

8.2 Computer Exercises

Computer Exercise 8.1

```
% ce8_1.m: BEPs for M-ary PSK in AWGN (uses upper bound for M > 4);
% coherent FSK (uses union bound); noncoherent FSK (exact)
%
clf
I_mod = input('Enter type of mod: MPSK = 1; CMFSK = 2; NCMFSK = 3: ');
if I_mod == 1
    z_dB = 3:.1:30;
elseif I_mod == 2 | I_mod == 3
    z_dB = 3:.1:15;
end
z = 10.^(z_dB/10);
for j = 1:6
    M=2^j;
     if I_mod == 1
        A=2/log2(M);
        kk=(sin(pi/M))^2*log2(M);
    elseif I_mod == 2
        A=M/2;
        kk=log2(M)/2;
    elseif I_mod == 3
        A=M/(2*(M-1));
        kk = 1;
     end
     if j == 1 & I_mod == 1
        Pb = .5*erfc(sqrt(z));
    elseif (j >= 2 & I_mod == 1) | (j >= 1 & I_mod == 2)
        Pb = (A/2)*erfc(sqrt(kk*z));
    elseif I_mod == 3
        Pb = zeros(size(z));
        for k = 1:M-1
            B = (prod(1:(M-1))/(prod(1:k)*prod(1:(M-1-k))))*(-1)^(k+1)/(k+1);
            alpha=k*log2(M)/(k+1);
            Pb = Pb+A*B*exp(-alpha*z);
        end
     end
    semilogy(z_dB,Pb),xlabel('E_b/N_0, dB'),ylabel('P_b'),...
```

```
            axis([min(z_dB) max(z_dB) 10^(-6) 1]),...
        if I_mod == 1
            title('BEP for MPSK')
            if j <= 2
                text(z_dB(35)+.2, Pb(35), 'M = 2 & 4')
            elseif j >= 3
                text(z_dB(105)+.2, Pb(105), ['M = ', num2str(M)])
            end
        elseif I_mod == 2
            title('BEP for coherent MFSK')
            if j <= 3
                text(z_dB(50)+.2, Pb(50), ['M = ', num2str(M)])
            elseif j > 3
                text(z_dB(25)+.2, Pb(25), ['M = ', num2str(M)])
            end
        elseif I_mod == 3
            title('BEP for noncoherent MFSK')
            if j <= 3
                text(z_dB(50)+.2, Pb(50), ['M = ', num2str(M)])
            elseif j > 3
                text(z_dB(25)+.2, Pb(25), ['M = ', num2str(M)])
            end
        end
        if j==1
            hold on
            grid
        end
    end
```

A typical run follows, with the resulting plot shown in Fig. 8.9:

```
>> ce8_1
Enter type of modulation: MPSK = 1; coh. MFSK = 2; noncoh. MFSK = 3: 1
```

Computer Exercise 8.2

```
% ce8_2.m: Out-of-band power for M-ary PSK, QPSK (OQPSK), and MSK
%
clf
A = char('-','–','-.',':','–.','-..');
Tbf = 0:.025:7;
```

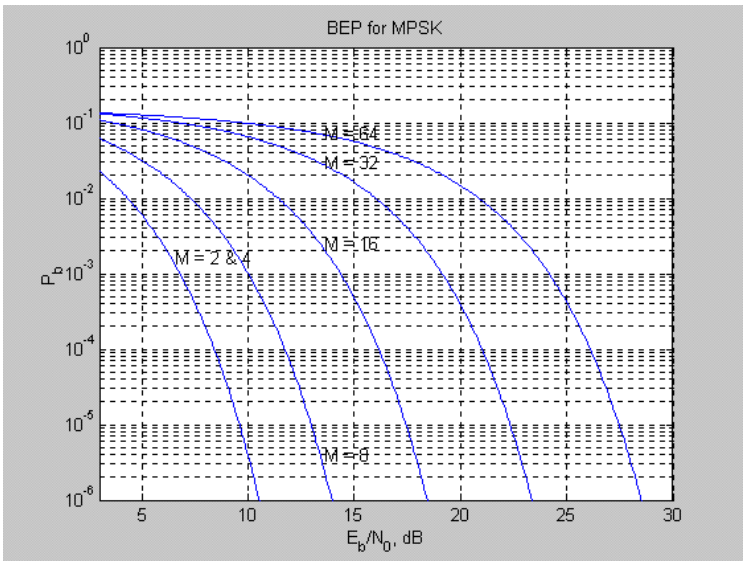

Figure 8.9:

```
LTbf = length(Tbf);
A_BPSK = [];
A_QPSK = [];
A_MSK = [];
for k = 1:LTbf
    A_BPSK(k) = quadl('G_BPSK', Tbf(k), 14);
    A_QPSK(k) = quadl('G_QPSK', Tbf(k), 7);
    A_MSK(k) = quadl('G_MSK', Tbf(k), 7);
end
OBP_BPSK = 10*log10(A_BPSK/A_BPSK(1));
OBP_QPSK = 10*log10(A_QPSK/A_QPSK(1));
OBP_MSK = 10*log10(A_MSK/A_MSK(1));
plot(Tbf, OBP_BPSK, A(1,:)), ylabel('Relative out-of-band power, dB'), xlabel('T_bf'),...
    axis([0 5 -40 0]),grid,...
    hold on
plot(Tbf, OBP_QPSK, A(2,:))
plot(Tbf, OBP_MSK, A(3,:))
legend('BPSK', 'QPSK/OQPSK', 'MSK')
```

A typical plot generated by the program is shown in Fig. 8.10.

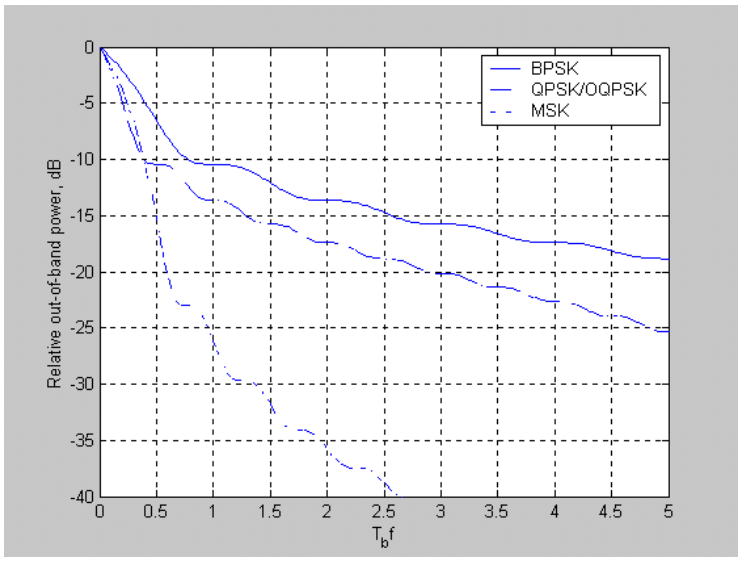

Figure 8.10:

Computer Exercise 8.3

```
% ce8_3.m: Computes spectrum of continuous phase FSK
%
clf
A = char('-','–','-.',':','–.','-..');
Tbf0 = input('Enter f_0 times T_b ')
delTbf0 = .5:.25:1.5;
L_delTbf = length(delTbf0)
Tbf = 0:.025:10;
for k = 1:L_delTbf
    delTbf = delTbf0(k);
    G_FSK = (S_CFSK(Tbf-Tbf0) + S_CFSK(Tbf-Tbf0-delTbf)).^2;
    area = sum(G_FSK)*0.025;
    G_FSKN = G_FSK/area;
    plot(Tbf, G_FSKN, A(k,:))
    if k == 1
        hold on
        grid
        xlabel('T_bf'), ylabel('Spectral amplitude')
    end
```

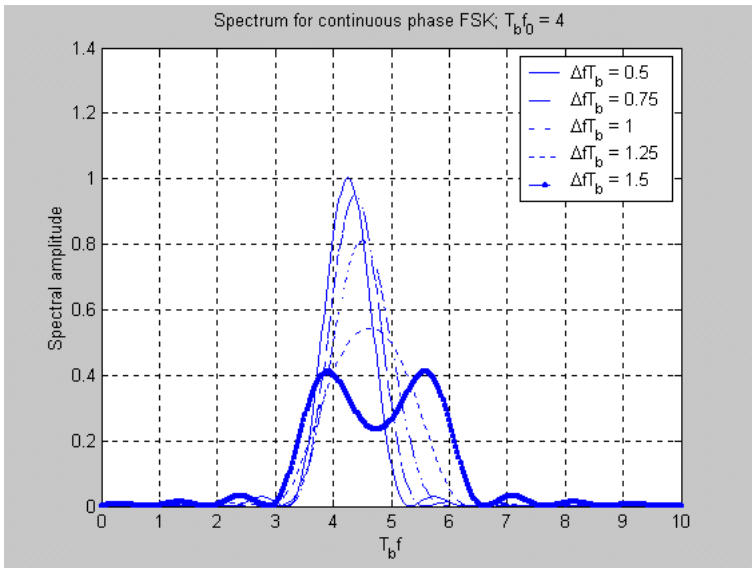

Figure 8.11:

```
    end
legend(['\DeltafT_b = ',num2str(delTbf0(1))], ['\DeltafT_b = ',num2str(delTbf0(2))],
['\DeltafT_b = ',num2str(delTbf0(3))],['\DeltafT_b = ',num2str(delTbf0(4))],
['\DeltafT_b = ',num2str(delTbf0(5))])
title(['Spectrum for continuous phase FSK; T_bf_0 = ', num2str(Tbf0)])

% Function to approximate CFSK spectrum
%
function y = S_CFSK(Tbf)
y = sinc(Tbf);
```

A typical plot is shown in Fig. 8.11.

Computer Exercise 8.4

```
% ce8_4.m: Computes bit error probability curves for jamming in DSSS.
% For a desired P_E and given JSR and Gp, computes the required Eb/N0
%
clf
A = char('-','–','-.',':','–.','-.x','-.o');
Gp_dB = input('Enter desired processing gain in dB ');
```

```
Gp = 10^(Gp_dB/10);
z_dB = 0:.5:30;
z = 10.^(z_dB/10);
k = 1;
JSR0 = [];
for JSR_dB = 5:5:25;
    JSR0(k) = JSR_dB;
    JSR = 10^(JSR_dB/10);
    arg = z./(1+z*JSR/Gp);
    PE = 0.5*erfc(sqrt(arg));
    semilogy(z_dB, PE, A(k+1,:))
    if k == 1
        hold on
        axis([0 30 1E-15 1])
        grid on
        xlabel('E_b/N_0, dB'), ylabel('P_E')
        title(['BEP for DS BPSK in jamming for proc. gain = ',
                    num2str(Gp_dB),' dB'])
    end
    k = k+1;
end
PEG = 0.5*qfn(sqrt(2*z));
semilogy(z_dB, PEG), text(z_dB(30)-5, PEG(30), 'No jamming')
legend(['JSR = ', num2str(JSR0(1))], ['JSR = ', num2str(JSR0(2))],
['JSR = ', num2str(JSR0(3))], ['JSR = ', num2str(JSR0(4))],
['JSR = ', num2str(JSR0(5))], 3)
disp('Strike ENTER to continue');
pause
PE0 = input('Enter desired value of P_E ');
JSR_dB_0 = input('Enter given value of JSR in dB ');
Gp_dB_0 = input('Enter given value of processing gain in dB ');
JSR0 = 10^(JSR_dB_0/10);
Gp0 = 10^(Gp_dB_0/10);
PELIM = 0.5*erfc(sqrt(Gp0/JSR0));
disp(' ')
disp('Infinite Eb/N0 BEP limit')
disp(PELIM)
 if PELIM >= PE0
    disp('For given choices of G_p, JSR, & desired P_E,
                                a solution is not possible ');
```

```
        Gp0_over_JSR0 = (erfinv(1-2*PE0))^2;
        Gp0_over_JSR0_dB = 10*log10(Gp0_over_JSR0);
        disp('The minimum required value of G_p over JSR in dB is:')
        disp(Gp0_over_JSR0_dB)
    else
        arg0 = (erfinv(1-2*PE0))^2;
        SNR0 = 1/(1/arg0 - JSR0/Gp0);
        SNR0_dB = 10*log10(SNR0);
    end
    disp(' ')
    disp('To give BEP of:')
    disp(PE0)
    disp('Requires GP, JSR, and Eb/N0 in dB of:')
    disp([Gp_dB JSR_dB SNR0_dB])
```

A typical run follows, with both a plot given in Fig. 8.12 and a specific output generated:

```
>> ce8_4
Enter desired processing gain in dB: 30
Strike ENTER to continue
Enter desired value of P_E: 1e-3
Enter given value of JSR in dB: 20
Enter given value of processing gain in dB: 30
Infinite Eb/N0 BEP limit
3.8721e-006
To give BEP of:
0.0010
Requires GP, JSR, and Eb/N0 in dB of:
30.0000 25.0000 9.6085
```

Computer Exercise 8.5

```
% ce8_5.m: Given a satellite altitude and desired spot diameter for
% illumination, determines circular antenna aperture diameter and
% maximum antenna gain.
%
h = input('Enter satellite altitude in km: ');
d_spot = input('Enter desired illuminated spot diameter at subsatellite
                                                        point in km: ');
f0 = input('Enter carrier frequency in GHz: ');
rho = input('Enter desired antenna efficiency: 0 < \rho <= 1: ');
```

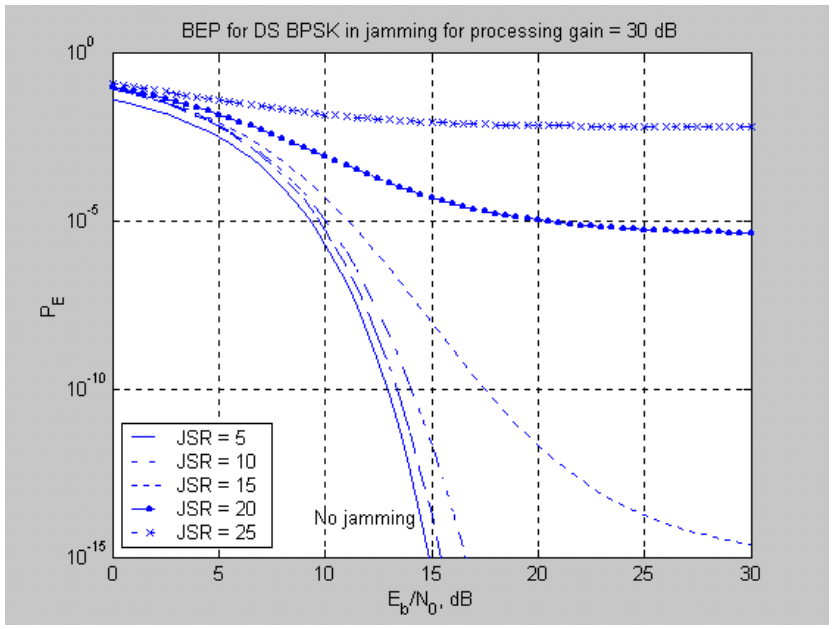

Figure 8.12:

```
theta = d_spot/h;
phi3dB = theta;
lambda = 3E8/(f0*1E9);
d = lambda/(phi3dB*sqrt(rho));
% G0 = rho*(pi*d/lambda)^2
G0 = (pi/phi3dB)^2;
G0_dB = 10*log10(G0);
disp(' ')
disp('3-dB beamwidth in degrees: ')
disp(phi3dB*57.3)
disp('Wavelength in meters:')
disp(lambda)
disp('Antenna diameter in meters:')
disp(d)
disp('Antenna gain in dB:')
disp(G0_dB)
disp(' ')
```

A typical run follows:

```
>> ce8_5
Enter satellite altitude in km: 200
Enter desired illuminated spot diameter at subsatellite point in km: 20
Enter carrier frequency in GHz: 8
Enter desired antenna efficiency: 0 < rho <= 1: .7
3-dB beamwidth in degrees:
5.7300
Wavelength in meters:
0.0375
Antenna diameter in meters:
0.4482
Antenna gain in dB:
29.9430
```

Computer Exercise 8.6

```
% ce8_6.m: Performance curves for bent-pipe relay and mod/demod relay
%
clf
A = char('–','-','-.',':');
I_mod = input('Enter type of modulation: 1 = BPSK; 2 = coh. FSK; 3 = DPSK;
                                                        4 = noncoh. FSK ');
PE0 = input('Enter desired value of P_E ');
for I_type = 1:2
    if I_type == 1
        if I_mod == 1
            EbN0r = (erfinv(1 - 2*PE0))^2;
        elseif I_mod == 2
            EbN0r = 2*(erfinv(1 - 2*PE0))^2;
        elseif I_mod == 3
            EbN0r = -log(2*PE0);
        elseif I_mod == 4
            EbN0r = -2*log(2*PE0);
        end
        EbN0r_dB = 10*log10(EbN0r)
        EbN0u_dB = EbN0r_dB+.000001:.01:35;
        EbN0u = 10.^(EbN0u_dB/10);
        den = 1/EbN0r-1./EbN0u;
        EbN0d = 1./den;
        EbN0d_dB = 10.*log10(EbN0d);
```

```
        elseif I_type == 2
            d1 = log10(0.00000001*PE0);
            d2 = log10(0.9999999*PE0);
            pu = logspace(d1, d2, 5000);
            pd = (PE0 - pu)./(1 - 2*pu);
             if I_mod == 1
                EbN0u = (erfinv(1 - 2*pu)).^2;
                EbN0d = (erfinv(1 - 2*pd)).^2;
            elseif I_mod == 2
                EbN0u = 2*(erfinv(1 - 2*pu)).^2;
                EbN0d = 2*(erfinv(1 - 2*pd)).^2;
            elseif I_mod == 3
                EbN0u = -log(2*pu);
                EbN0d = -log(2*pd);
            elseif I_mod == 4
                EbN0u = -2*log(2*pu);
                EbN0d = -2*log(2*pd);
            end
            EbN0u_dB = 10*log10(EbN0u);
            EbN0d_dB = 10*log10(EbN0d);
        end
        plot(EbN0d_dB, EbN0u_dB,A(I_type,:)),
        if I_type == 1
            axis square, axis([5 30 5 30]), grid on,...
            xlabel('(E_b/N_0)_d, dB'), ylabel('(E_b/N_0)_u, dB')
            hold on
        end
    end
    legend(['Bent pipe'],['Demod/remod'],1)
    if I_mod == 1
        title(['Uplink E_b/N_0 versus downlink E_b/N_0, both in dB,
             to give P_E = ',num2str(PE0),'; BPSK modulation'])
    elseif I_mod == 2
        title(['Uplink E_b/N_0 versus downlink E_b/N_0, both in dB,
             to give P_E = ',num2str(PE0),'; coh. FSK modulation'])
    elseif I_mod == 3
        title(['Uplink E_b/N_0 versus downlink E_b/N_0, both in dB,
            to give P_E = ',num2str(PE0),'; DPSK modulation'])
    elseif I_mod == 4
        title(['Uplink E_b/N_0 versus downlink E_b/N_0, both in dB,
```

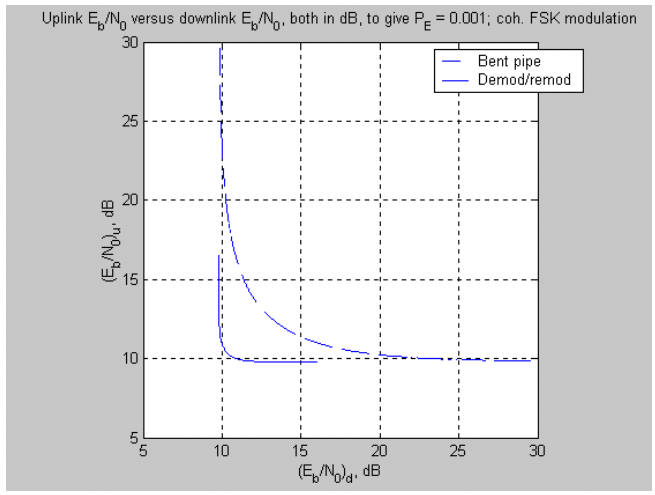

Figure 8.13:

```
            to give P_E = ',num2str(PE0),'; noncoh. FSK modulation'])
end
```

A typical run follows:

```
>> ce8_6
Enter type of modulation: 1 = BPSK; 2 = coh. FSK; 3 = DPSK; 4 = noncoh. FSK 2
Enter desired value of P_E 1e-3
EbN0r_dB =
9.7998
```

Computer Exercise 8.7

```
% ce8_7.m: plots waveforms or spectra for gmsk and msk
%
A = char('-','-.','–',':.');
samp_bit = input('Enter number of samples per bit used in simulation: ');
N_bits = input('Enter total number of bits: ');
GMSK = input('Enter 1 for GMSK; 0 for normal MSK: ');
if GMSK == 1
    BT_bit = input('Enter vector of bandwidth X bit periods: ');
    N_samp = 50;
end
```

```
I_plot = input('Enter 1 to plot waveforms; 2 to plot spectra and out-of-band power ');
if I_plot == 1
    f0 = 1;
elseif I_plot == 2
    f0 = 0;
end
clf
kp = pi/2;
T_bit = 1;
LB = length(BT_bit);
for k = 1:LB
    B = BT_bit(k)/T_bit;
    alpha = 1.1774/B; % Note that B is two-sided bandwidth
    del_t = T_bit/samp_bit;
    fs = 1/del_t;
    data = 0.5*(sign(rand(1,N_bits)-.5)+1);
    s = 2*data - 1;
    L = length(s);
    t=0:del_t:L*T_bit-del_t;
    s_t = s(ones(samp_bit,1),:); % Build array whose columns are samp_bit long
    s_t = s_t(:)'; % Convert matrix where bit samples occupy columns to vector
    L_s_t = length(s_t);
    tp=0:del_t:N_samp*T_bit-del_t;
    L_t = length(t);
    L_tp=length(tp);
    t_max = max(tp);
    if GMSK == 1
     hG = (sqrt(pi)/alpha)*exp(-pi^2*(tp-t_max/2).^2/alpha^2);
     freq1 = kp*del_t*conv(hG,s_t);
    L_hG = length(hG);
    freq = freq1(L_hG/2:length(freq1)-L_hG/2);
    elseif GMSK == 0
        freq = kp*s_t;
    end
    phase = del_t*cumsum(freq);
    if GMSK == 0
        y_mod = exp(j*2*pi*f0*t+j*phase);
    elseif GMSK == 1
         y_mod = exp(j*2*pi*f0*t+j*phase);
    end
```

```
if I_plot == 1
    subplot(3,1,1),plot(t,s_t),axis([min(t), max(t), -1.2, 1.2]), ylabel('Data')
    if GMSK == 0
        title('MSK waveforms')
    elseif GMSK == 1
        title(['GMSK for BT_b = ', num2str(BT_bit(k))])
    end
    subplot(3,1,2),plot(t,phase), ylabel('Excess phase, rad.')
    for nn = 1:11
         if nn == 1
             hold on
         end
        subplot(3,1,2),plot(t,-nn*(pi/2)*ones(size(t)),'m')
        subplot(3,1,2),plot(t,nn*(pi/2)*ones(size(t)),'m')
    end
    subplot(3,1,3),plot(t,real(y_mod)), xlabel('t'), ylabel('Modulated signal')
elseif I_plot == 2
    Z=PSD(real(y_mod),2048,fs);          % MATLAB supplied function
    ZN = Z/max(Z);
    ET = sum(ZN);
    OBP = 1-cumsum(ZN)/ET;
    LZ = length(Z);
    del_FR = fs/2/LZ;
    FR = [0:del_FR:fs/2-del_FR];
    subplot(1,2,1),semilogy(FR,ZN,A(k,:)), axis([0 3 10^(-8) 1])
     if k == 1
         hold on
         grid
         ylabel('PSD, J/Hz'), xlabel('fT_b_i_t')
        title('Power spectra and out-of-band power for GMSK')
    end
    subplot(1,2,2),semilogy(FR,OBP,A(k,:)), axis([0 3 10^(-8) 1])
    if k == 1
        hold on
        grid
        ylabel('Fraction of out-of-band power'), xlabel('fT_b_i_t')
    end
    if LB == 1
        legend(['BT_b_i_t = ',num2str(BT_bit(1))],1)
    elseif LB == 2
```

```
            legend(['BT_b_i_t = ',num2str(BT_bit(1))],
                   ['BT_b_i_t = ',num2str(BT_bit(2))],1)
        elseif LB == 3
            legend(['BT_b_i_t = ',num2str(BT_bit(1))],
                   ['BT_b_i_t = ',num2str(BT_bit(2))],
                   ['BT_b_i_t = ',num2str(BT_bit(3))],1)
        elseif LB == 4
            legend(['BT_b_i_t = ',num2str(BT_bit(1))],
                   ['BT_b_i_t = ',num2str(BT_bit(2))],
                   ['BT_b_i_t = ',num2str(BT_bit(3))],
                   ['BT_b_i_t = ',num2str(BT_bit(4))],1)
        end
    end
end
```

Two typical runs follow - one for plotting waveforms and one for plotting spectra.

```
>> ce8_7
Enter number of samples per bit used in simulation: 20
Enter total number of bits: 50
Enter 1 for GMSK; 0 for normal MSK: 1
Enter vector of bandwidth X bit periods: .5
Enter 1 to plot waveforms; 2 to plot spectra and out-of-band power: 1

>> ce8_7
Enter number of samples per bit used in simulation: 10
Enter total number of bits: 2000
Enter 1 for GMSK; 0 for normal MSK: 1
Enter vector of bandwidth X bit periods: [.5 1 1.5]
Enter 1 to plot waveforms; 2 to plot spectra and out-of-band power: 2
```

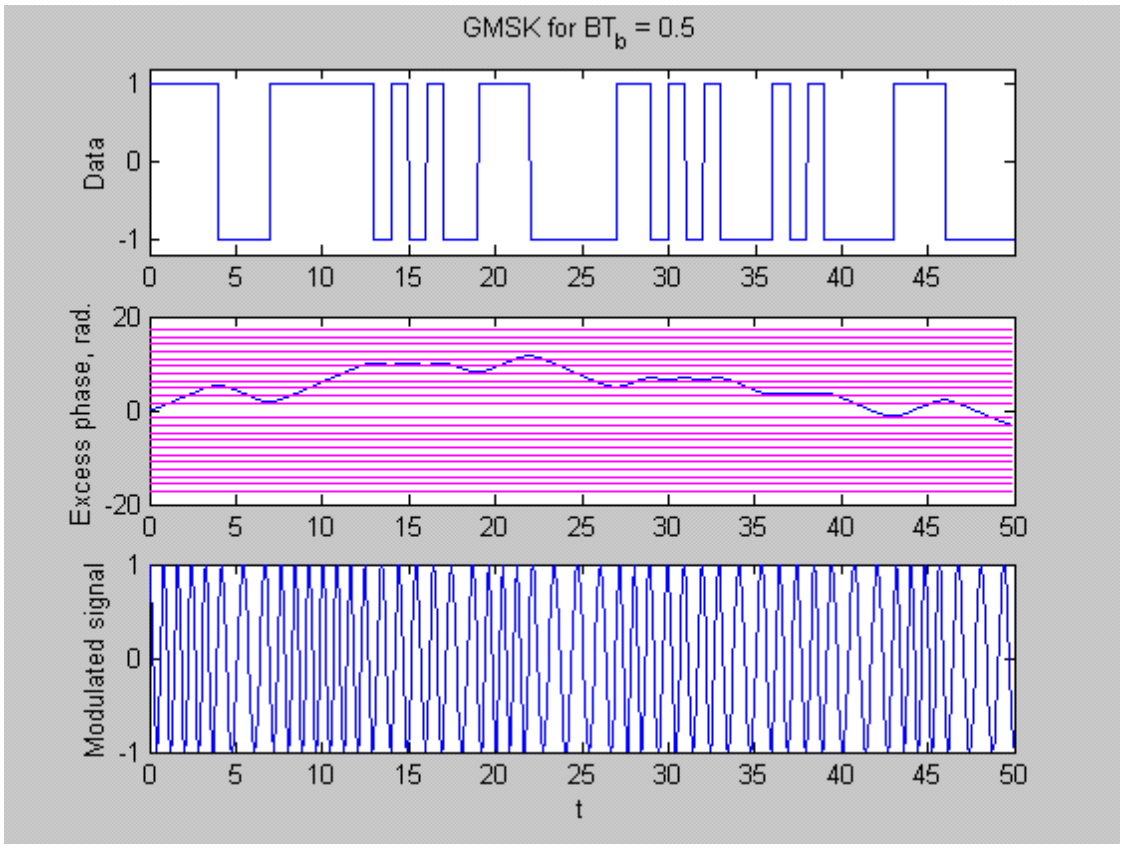
GMSK for BT$_b$ = 0.5
Data
1
0
-1
0 5 10 15 20 25 30 35 40 45
Excess phase, rad.
20
0
-20
0 5 10 15 20 25 30 35 40 45 50
Modulated signal
1
0
-1
0 5 10 15 20 25 30 35 40 45 50
t

Figure 8.14:

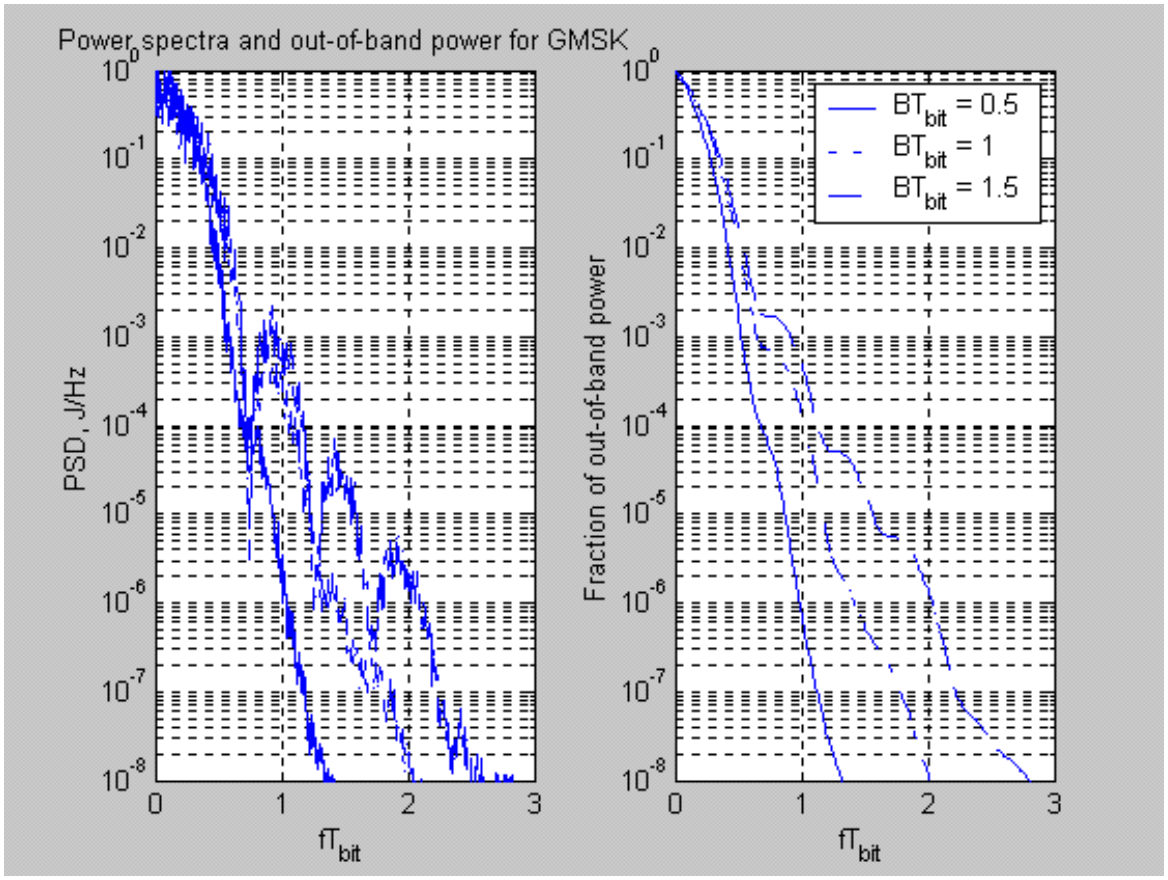
Power spectra and out-of-band power for GMSK
PSD, J/Hz
Fraction of out-of-band power
fT_bit
fT_bit
BT_bit = 0.5
BT_bit = 1
BT_bit = 1.5

Figure 8.15:

Chapter 9

Optimum Receivers and Signal Space Concepts

9.1 Problems

Problem 9.1

a. Given H_1, $Z = N$, so

$$f_Z(z|H_1) = f_N(n)|_{z=n} = 10e^{-10z}u(z)$$

Given H_2, $Z = S+N$, where S and N are independent. Thus, the resulting pdf under H_2 is the convolution of the separate pdfs of S and N:

$$f_Z(z|H_2) = \int_{-\infty}^{\infty} f_s(z-\lambda) f_N(\lambda) d\lambda = 20\int_0^z e^{-2z}e^{-8\lambda}d\lambda, \quad \lambda > 0$$

which integrates to the result given in the problem statement.

b. The likelihood ratio is

$$\Lambda(Z) = \frac{f_Z(Z|H_2)}{f_Z(Z|H_1)} = 0.25\left(e^{8Z}-1\right), \quad Z > 0$$

c. The threshold is

$$\eta = \frac{P_0(c_{21}-c_{11})}{P_1(c_{12}-c_{22})} = \frac{1}{3}$$

d. The likelihood ratio test becomes

$$0.25\left(e^{8Z}-1\right) \underset{H_1}{\overset{H_2}{\gtrless}} \frac{1}{3}$$

This can be reduced to

$$Z \underset{H_1}{\overset{H_2}{\gtrless}} \frac{\ln(7/3)}{8} = 0.106$$

e. The probability of detection is

$$P_D = \int_{0.106}^{\infty} 2.5\left(e^{-2z}-e^{-10z}\right)dz = 0.925$$

The probability of false alarm is

$$P_F = \int_{0.106}^{\infty} 10e^{-10z}dz = 0.346$$

Therefore, the risk is

$$\text{Risk } = \frac{1}{4}(5) + \frac{3}{4}(5)(1-0.925) - \frac{1}{4}(5)(1-0.346) = 0.714$$

f. Consider the threshold set at η. Then

$$P_F = e^{-10\eta} \Rightarrow \eta \geq 0.921$$

to give $P_F \leq 10^{-4}$. Also, from part (e),

$$P_D = 1.25e^{-2\eta} - 0.25e^{-10\eta}$$

For values of $\eta \geq 0.921$, P_D is approximately equal to the first term of the above expression. For this range of ηP_D is strictly decreasing with η , so the best value of P_D occurs for $\eta = 0.921$ for $P_F \leq 10^{-4}$. The resulting P_D is 0.198.

g. From part (f),

$$P_F = e^{-10\eta} \text{ and } P_D = 1.25e^{-2\eta} - 0.25e^{-10\eta}$$

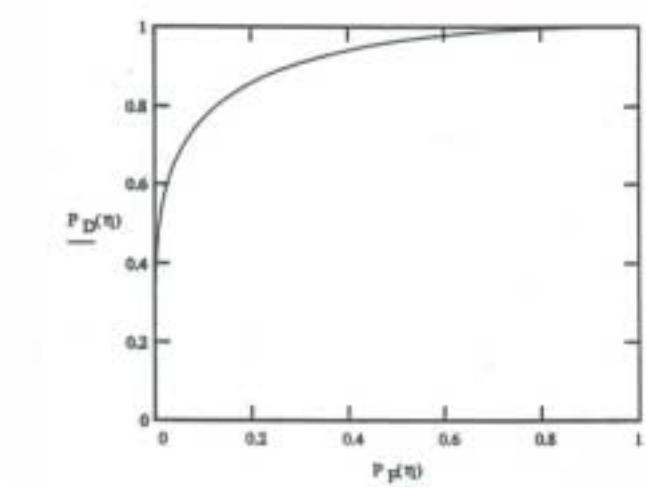

Figure 9.1:

From part (d), the likelihood ratio test is

$$Z \underset{H_1}{\overset{H_2}{\gtrless}} \frac{\ln(4\eta+1)}{8} = \gamma$$

A plot of P_D versus P_F as a function of η or γ constitutes the operating characteristic of the test. It is shown in Figure 9.1.

Problem 9.2

a. The likelihood ratio is

$$\Lambda(Z) = \frac{\frac{1}{2}e^{|Z|}}{\frac{e^{-\frac{1}{2}Z^2}}{\sqrt{2\pi}}} = \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}Z^2-|Z|}$$

b. The conditional pdfs under each hypothesis are plotted below. The decision regions R_1 and R_2 are indicated in Figure 9.2 for $\eta = 1$.

Problem 9.3

Define $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Define scalar multiplication by $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$ and vector addition by $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. The properties listed under "Structure of Signal Space" in the text then become:

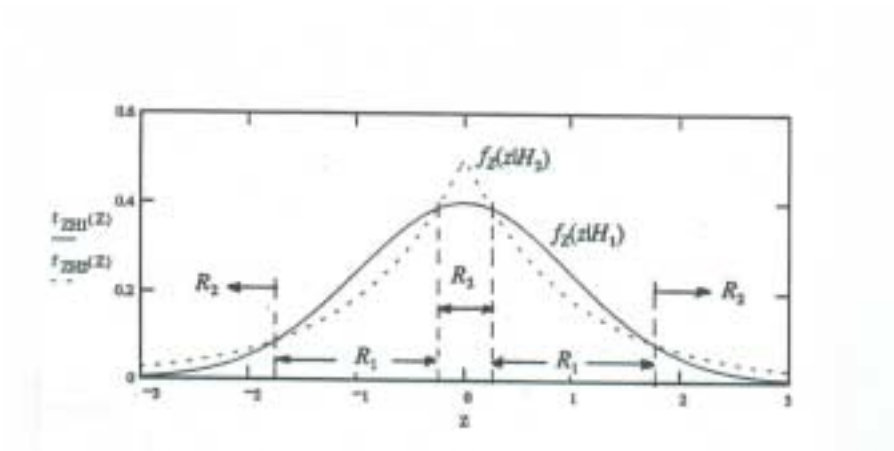

Figure 9.2:

1. $\alpha_1 A + \alpha_2 B = (\alpha_1 a_1 + \alpha_2 b_1, \alpha_1 a_2 + \alpha_2 b_2, \alpha_1 a_3 + \alpha_2 b_3)\, \epsilon R^3$;
2. $\alpha\,(A + B) = \alpha\,(a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$
   $= (\alpha a_1 + \alpha b_{1,} \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3) = (\alpha a_1, \alpha a_{2,} \alpha a_3) + (\alpha b_{1,} \alpha b_2, \alpha b_3)$
   $= \alpha A + \alpha B\, \epsilon\, R^3$;
3. $\alpha_1\,(\alpha_2 A) = (\alpha_1 \alpha_2 A)\ \epsilon\, R^3$ (follows by writing out in component form);
4. $1 \bullet A = A$ (follows by writing out in component form);
5. The unique element 0 is $(0, 0, 0)$ so that $A + 0 = A$;
6. $-A = (-a_1, -a_{2,} - a_3)$ so that $A + (-A) = 0$.

Problem 9.4
Consider

$$(x, y) = \lim_{T \to \infty} \int_{-T}^{T} x\,(t)\, y * (t)\, dt$$

Then

$$(x, y) = \lim_{T \to \infty} \int_{-T}^{T} y\,(t)\, x^*\,(t)\, dt = \left[\lim_{T \to \infty} \int_{-T}^{T} x\,(t)\, y^*\,(t)\, dt \right]^* = (x, y)^*$$

Also

$$(\alpha x, y) = \lim_{T \to \infty} \int_{-T}^{T} \alpha x\,(t)\, y^*\,(t)\, dt = \alpha \lim_{T \to \infty} \int_{-T}^{T} x\,(t)\, y^*\,(t)\, dt = \alpha\,(x, y)$$

and

$$\begin{aligned}(x+y,z) &= \lim_{T\to\infty}\int_{-T}^{T}\left[x(t)+y(t)\right]z(t)^{*}(t)\,dt \\ &= \lim_{T\to\infty}\int_{-T}^{T}x(t)\,z^{*}(t)\,dt+\lim_{T\to\infty}\int_{-T}^{T}y(t)\,z^{*}(t)\,dt \\ &= (x,z)+(y,z)\end{aligned}$$

Finally

$$(x,x)=\lim_{T\to\infty}\int_{-T}^{T}x(t)\,x^{*}(t)\,dt=\lim_{T\to\infty}\int_{-T}^{T}\left|x(t)\right|^{2}dt\geq 0$$

The scalar product for power signals is considered in the same manner.

Problem 9.5

a. This scalar product is

$$(x_1,x_2)=\lim_{T\to\infty}\int_{0}^{T}2e^{-6t}dt=\frac{1}{3}$$

b. Use the energy signal scalar product definition as in (a):

$$(x_1,x_2)=\lim_{T\to\infty}\int_{0}^{T}e^{-(5+j)t}dt=\frac{1}{5+j}$$

c. Use the power signal product definition:

$$(x_1,x_2)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(2\pi t)\sin^{2}(2\pi t)\,dt=0$$

d. Use the power signal scalar product definition:

$$(x_1,x_2)=\lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T}\cos(2\pi t)\,dt=0$$

Problem 9.6

Since the signals are assumed real,

$$\begin{aligned}\|x_1+x_2\|^{2} &= \lim_{T\to\infty}\int_{-T}^{T}\left[x_1(t)+x_2(t)\right]^{2}dt \\ &= \lim_{T\to\infty}\int_{-T}^{T}x_1^{2}(t)\,dt+2\lim_{T\longrightarrow\infty}\int_{-T}^{T}x_1(t)\,x_2(t)\,dt+\lim_{T\longrightarrow\infty}\int_{-T}^{T}x_2^{2}dt \\ &= \|x_1\|^{2}+2(x_1,x_2)+\|x_2\|^{2}\end{aligned}$$

From this it is seen that

$$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$$

if and only if $(x_1, x_2) = 0$.

Problem 9.7

By straight forward integration, it follows that

$$\begin{aligned} \|x_1\|^2 &= 2 \quad \text{or} \quad \|x_1\| = \sqrt{2} \\ \|x_2\|^2 &= \frac{2}{3} \quad \text{or} \quad \|x_2\| = \sqrt{\frac{2}{3}} \\ \|x_3\|^2 &= \frac{8}{3} \quad \text{or} \quad \|x_3\| = \sqrt{\frac{8}{3}} \end{aligned}$$

Also

$$(x_1, x_2) = 0 \text{ and } (x_3, x_1) = 2$$

Since $(x_1, x_2) = 0$, they are orthogonal. Choose them as basis functions (not normalized). Clearly, x_3 is their vector sum.

Problem 9.8

It follows that

$$\|x_1\|^2 = \sum_{n=1}^{N} |a_n|^2 \text{ and } \|x_2\|^2 = \sum_{n=1}^{N} |b_n|^2$$

Also

$$(x_1, x_2) = \sum_{n=1}^{N} a_n b_n^*$$

The inequality can be reduced to

$$\sum_m \sum_n |a_n b_m - a_m b_n|^2 \geq 0$$

which is true because each term in the sum is nonnegative.

Problem 9.9

a. A set of normalized basis functions is

$$\begin{aligned} \phi_1(t) &= \frac{1}{\sqrt{2}} s_1(t) \\ \phi_2(t) &= \sqrt{\frac{2}{3}} \left[s_2(t) - \frac{1}{2} s_1(t) \right] \\ \phi_3(t) &= \sqrt{3} \left[s_3(t) - \frac{2}{3} s_1(t) - \frac{2}{3} s_2(t) \right] \end{aligned}$$

b. Evaluate (s_1, ϕ_1), (s_2, ϕ_1), and (s_3, ϕ_1) to get

$$s_1 = \sqrt{2}\phi_1;\ s_2 = \frac{1}{\sqrt{2}}\phi_1 + \sqrt{\frac{3}{2}}\phi_2;\ s_3 = \sqrt{2}\phi_1 + \sqrt{\frac{2}{3}}\left(\phi_2 + \phi_3\right)$$

Note that a basis function set could have been obtained by inspection. For example, three unit-height, unit-width, nonoverlapping pulses spanning the interval could have been used.

Problem 9.10
A basis set is

$$\phi_1(t) = Ks_1(t) \text{ and } \phi_2(t) = Ks_2(t)$$

where

$$K = \sqrt{\frac{f_c}{NA^2}}$$

A signal point lies on each coordinate axis at $\pm 1/K$.

Problem 9.11
First we set

$$v_1(t) = x_1(t) = \exp(-t)u(t)$$

By definition

$$\|v_1\|^2 = \|x_1\|^2 = \int_0^\infty \exp(-2t)dt = \frac{1}{2}$$

Thus

$$\|v_1\| = \frac{1}{\sqrt{2}}$$

This leads to the first basis function

$$\phi_1(t) = \sqrt{2}\exp(-t)u(t)$$

For the second basis function we compute <1

$$v_2(t) = x_2(t) - (x_2, \phi_1)\phi_1(t)$$

where

$$(x_2, \phi_1) = \int_0^\infty \sqrt{2}\exp(-t)\exp(-2t)dt = \sqrt{2}\int_0^\infty \exp(-3t)dt = \frac{\sqrt{2}}{3}$$

Thus

$$v_2(t) = \exp(-2t)u(t) - \frac{\sqrt{2}}{3}\sqrt{2}\exp(-t)u(t) = \left[\exp(-2t) - \frac{2}{3}\exp(-t)\right]u(t)$$

Next we compute

$$\|v_2\|^2 = \int_0^\infty \left(\exp(-4t) - \frac{4}{3}\exp(-3t) + \frac{4}{9}\exp(-2t)\right) dt = \frac{1}{36}$$

This gives

$$\phi_2(t) = [6\exp(-2t) - 4\exp(-t)]\, u(t)$$

We next consider

$$v_3(t) = x_3(t) - (x_3, \phi_2)\phi_2(t) - (x_3, \phi_1)\phi_1(t)$$

Using the same procedure as previously used

$$(x_3, \phi_2) = \int_0^\infty \exp(-3t)\,[6\exp(-2t) - 4\exp(-t)]\, dt = \frac{1}{5}$$

so that

$$(x_3, \phi_2)\phi_2(t) = \frac{6}{5}\exp(-2t) - \frac{4}{5}\exp(-t)$$

Also

$$(x_3, \phi_1) = \int_0^\infty \exp(-3t)\sqrt{2}\exp(-t)dt = \frac{\sqrt{2}}{4}$$

so that

$$(x_3, \phi_1)\phi_1(t) = \frac{1}{2}\exp(-t)$$

For $t \geq 0$ we then have

$$v_3(t) = \exp(-3t) - \frac{6}{5}\exp(-2t) + \frac{3}{10}\exp(-t)$$

This leads to

$$\|v_3\|^2 = \frac{229}{600}$$

and

$$\phi_3(t) = \sqrt{\frac{600}{229}}\left[\exp(-3t) - \frac{6}{5}\exp(-2t) + \frac{3}{10}\exp(-t)\right] u(t)$$

Problem 9.12

First we set

$$v_1(t) = x_1(t) = t$$

By definition

$$\|v_1\|^2 = \|x_1\|^2 = \int_{-1}^1 t^2 dt = \frac{2}{3}$$

Thus the first basis function is

$$\phi_1(t) = \sqrt{\frac{3}{2}}t, \qquad -1 < t < 1$$

For the next basis function we set

$$v_2(t) = x_2(t) - (x_2, \phi_1)\phi_1(t)$$

where

$$(x_2, \phi_1) = \int_{-1}^{1} t^2 \left(\sqrt{\frac{3}{2}}t\right) dt = 0$$

since the intergrand is odd and the limits are even. Thus

$$\|v_2\|^2 = \int_{-1}^{1} t^4 dt = \frac{2}{5}$$

from which

$$\phi_2(t) = \frac{t^2}{\|v_2\|} = \sqrt{\frac{5}{2}}t^2$$

For the third basis function we write

$$v_3(t) = x_3(t) - (x_3, \phi_2)\phi_2(t) - (x_3, \phi_1)\phi_1(t)$$

where

$$(x_3, \phi_2) = \int_{-1}^{1} t^3 \left(\sqrt{\frac{5}{2}}t^2\right) dt = 0$$

and

$$(x_3, \phi_1) = \int_{-1}^{1} t^3 \left(\sqrt{\frac{3}{2}}t\right) dt = \sqrt{\frac{3}{2}}\frac{2}{5}$$

so that

$$v_3(t) = t^3 - \sqrt{\frac{3}{2}}\frac{2}{5}\sqrt{\frac{3}{2}}t = t^3 - \frac{3}{5}t$$

This gives

$$\|v_3\|^2 = \int_{-1}^{1} \left[t^2 - \frac{6}{5}t^4 + \frac{9}{25}t^2\right] dt = \frac{8}{175}$$

and

$$\phi_3(t) = \frac{v_3(t)}{\|v_3\|} = \sqrt{\frac{175}{8}}\left[t^3 - \frac{3}{5}t\right]$$

For the next basis function we write

$$v_4(t) = x_4(t) - (x_4, \phi_3)\phi_3(t) - (x_4, \phi_2)\phi_2(t) - (x_4, \phi_1)\phi_1(t)$$

or

$$v_4(t) = x_4(t) - (x_4, \phi_2)\phi_2(t)$$

since (x_4, ϕ_3) and (x_4, ϕ_1) are zero. By definition

$$(x_4, \phi_2) = \int_{-1}^{1} t^4 \left(\sqrt{\frac{5}{2}}t^2\right) dt = \sqrt{\frac{5}{2}}\frac{2}{7}$$

so that

$$v_4(t) = t^4 - \sqrt{\frac{5}{2}}\frac{2}{7}\sqrt{\frac{5}{2}}t^2 = t^4 - \frac{5}{7}t^2$$

This gives

$$\|v_4\|^2 = \int_{-1}^{1} \left[t^8 - \frac{10}{7}t^6 + \frac{25}{49}t^4\right] dt = \frac{8}{441}$$

and

$$\phi_4(t) = \frac{v_4(t)}{\|v_4\|} = \sqrt{\frac{441}{8}}\left[t^4 - \frac{5}{7}t^2\right]$$

Problem 9.13

a. A suitable basis is

$$\begin{aligned} \phi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \text{ and } \phi_2(t) \\ &= \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \text{ for } 0 \le t \le T \end{aligned}$$

The coordinates of the signal vectors are

$$x_i = \int_0^T s_i(t)\,\phi_1(t)\,dt = \sqrt{E}\cos\left(\frac{2\pi i}{M}\right) = \sqrt{E}\cos\psi_i$$

and

$$y_i = \int_0^T s_i(t)\,\phi_2(t)\,dt = -\sqrt{E}\sin\left(\frac{2\pi i}{M}\right) = -\sqrt{E}\sin\psi_i$$

where $E = A^2T/2$ and $\psi_i = \frac{2\pi i}{M}$. Thus

$$s_i(t) = \sqrt{E}\cos(\psi_i)\phi_1(t) - \sqrt{E}\sin(\psi_i)\phi_2(t), \quad i = 1, 2, 3, 4$$

b. The optimum partitioning of the signal space is a pie-wedge centered on each signal point with the origin as the vertex and having angle $2\pi/M$.

c. The optimum receiver consists of two parallel branches which correlate the incoming signal plus noise with $\phi_1(t)$ and $\phi_2(t)$. The correlator outputs are sampled at the end of each signaling interval. Call these samples x and y. The angle $\tan^{-1}(x/y)$ is formed to determine which pie-wedge decision region the received data vector is in.

d. Defining $z_i(t) = s_i(t) + n(t)$ gives

$$z_i(t) = (\sqrt{E}\cos(\psi_i) + N_1)\phi_1(t) + (-\sqrt{E}\sin(\psi_i) + N_2)\phi_2(t), \quad i = 1, 2, 3, 4$$

The probability of error can then be written in the form

$$\begin{aligned} P[\text{error}\,|s_i(t)] &= 1 - \Pr[\text{correct recept.}|s_i(t)] \\ &= 1 - \int\int_{R_i} \frac{1}{\pi N_0} \\ &\quad \cdot \exp\left[-\frac{1}{N_0}\left(\left(x - \sqrt{E}\cos\psi_i\right)^2 + \left(y + \sqrt{E}\sin(\psi_i)\right)^2\right)\right] dxdy \end{aligned}$$

where R_i is the region for a correct decision given that $s_i(t)$ is transmitted. (Note that the noise variance is $N_0/2$). The expression for the probability of error can be changed to polar coordinates by letting

$$x = r\sqrt{N_0}\cos\theta \text{ and } y = r\sqrt{N_0}\sin\theta$$

With this transformation

$$dxdy \to N_0 r dr d\theta$$

This gives

$$\begin{aligned} P[\text{error}\,|s_i(t)] &= 1 - \frac{1}{\pi}\int\int_{R_i} r \\ &\quad \cdot \exp\left[-\frac{1}{N_0}\left(\left(r^2 N_0\right) - 2r\sqrt{EN_0}\cos(\theta + \psi_i) + E\right)\right] dr d\theta \end{aligned}$$

where R_i now represents the decision region for a correct decision in the R, θ plane. Since all decision regions have equal area and the noise has circular symmetry, the error probability is independent of the transmitted signal. Thus, let $i = M$ so that $\psi_i = 2\pi$ and $\cos(\theta + \psi_i) = \cos\theta$. Since the error probability is independent of the signal chosen, the conditional symbol error probability is equal to the unconditional error probability for the case in which all signals are transmitted with equal probability. Therefore

$$P[\text{error}] = 1 - \frac{1}{\pi}\int_{-\pi/M}^{\pi/M}\int_0^\infty r\exp\left[-\left(r^2 - 2r\sqrt{E/N_0}\cos(\theta) + E/N_0\right)\right] dr d\theta$$

e. This was done in Chapter 8.

Problem 9.14
Write

$$P_c = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{\sqrt{\pi}} \left[\int_{-\infty}^{y+\sqrt{E/N_0}} \frac{e^{-x^2}}{\sqrt{\pi}} dx \right] dy$$

and let $z = x - y$. This gives

$$P_c = \int_{-\infty}^{\sqrt{E/N_0}} \frac{e^{-z^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-2(y+z/2)^2}}{\sqrt{\pi}} dydz$$

Complete the square in the exponent of the inside integral and use a table of definite integrals to show that is evaluates to $(1/2)^{1/2}$. The result then reduces to

$$P_c = 1 - Q\left(\sqrt{E/N_0}\right)$$

which gives $P_c = Q\left[(E/N_0)^{1/2}\right]$, the desired result.

Problem 9.15

a. The space is three-dimensional with signal points at the eight points $\left(\pm\sqrt{E/3}\right)$, $\left(\pm\sqrt{E/3}\right)$, $\left(\pm\sqrt{E/3}\right)$. The optimum partitions are planes determined by the coordinate axes taken two at a time (three planes). Thus the optimum decision regions are the eight quadrants of the signal space.

b. Consider S_1. Given the partitioning discussed in part (a), we make a correct decision only if

$$\|Y - S_1\|^2 < \|Y - S_2\|^2$$

and

$$\|Y - S_1\|^2 < \|Y - S_4\|^2$$

and

$$\|Y - S_1\|^2 < \|Y - S_8\|^2$$

where S_2, S_4, and S_8 are the nearest-neighbor signal points to S_1. These are the most probable errors. Substituting $Y = (y_1, y_2, y_3)$ and $S_1 = (1, 1, 1)$, $S_2\,(1, -1, 1,)$, $S_4\,(-1, 1, 1)$, and $S_8 = (1, 1, -1)$, the above conditions become

$$(y_1 + 1)^2 < (y_1 - 1)^2, (y_2 + 1)^2 < (y_2 - 1)^2, \text{ and } (y_3 + 1)^2 < (y_3 - 1)^2$$

by canceling like terms on each side. For S_1, these reduce to

$$y_1 > 0, y_2 > 0, \text{ and } y_3 > 0$$

to define the decision region for S_1. Therefore, the probability of correct decision is

$$P\left[\text{correct dec.}|s_1(t)\right] = \Pr\left(y_1 > 0,\ y_2 > 0,\ y_3 > 0\right) = \left[\Pr\left(y_i > 0\right)\right]^3$$

because the noises along each coordinate axis are independent. Note that $E\left[y_i\right] = (E/3)^{1/2}$, all i. The noise variances, and therefore the variances of the y_i's are all N_0. Thus

$$P\left[\text{correct dec.}|s_1(t)\right] = \left[\frac{1}{\sqrt{\pi N_0}}\int_0^\infty e^{-\frac{1}{N_0}\left(y-\sqrt{E/3}^2\right)}dy\right]^3 = \left[1 - Q\left(\sqrt{2E/3N_0}\right)\right]^3$$

Since this is independent of the signal chosen, this is the average probability of correct detection. The symbol error probability is $1 - P_c$. Generalizing to n dimensions, we have

$$P_e = 1 - \left[1 - Q\left(\sqrt{2E/nN_0}\right)\right]^n$$

where $n = \log_2 M$. Note that $E_b = E/n$ since there are n bits per dimension.

c. The symbol error probability is plotted in Figure 9.3.

Problem 9.16

a. We need $1/2T$ Hz per signal. Therefore, $M = W/\left(1/2T\right) = 2WT$.

b. For vertices-of-a-hpercube signaling, $M = 2^n$ where n is the number of dimensions. Hence $M = 2^{2WT}$.

Problem 9.17

In the equation

$$P\left(E|H_1\right) = \int_0^\infty \left[\int_{r_1}^\infty f_{R_2}\left(r_2|H_1\right)dr_2\right] f_{R_1}\left(r_1|H_1\right)dr_1$$

substitute

$$f_{R_1}\left(r_1|H_1\right) = \frac{2r_1}{2E\sigma^2 + N_0}\exp\left(-\frac{r_1^2}{2E\sigma^2 + N_0}\right); \qquad r_1 \geq 0$$

and

$$f_{R_2}\left(r_2|H_1\right) = \frac{2r_2}{N_0}\exp\left(-\frac{r_2^2}{N_0}\right), \qquad r_2 > 0$$

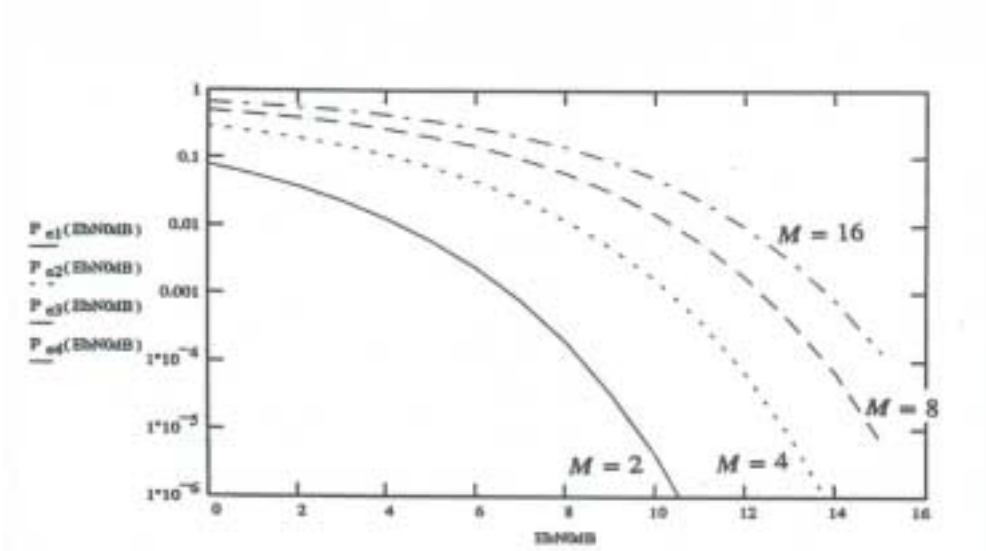

Figure 9.3:

The inside integral becomes

$$I = \exp\left(-\frac{r_1^2}{N_0}\right)$$

When substituted in the first equation, it can be reduced to

$$P\left(E|H_1\right) = \int_0^\infty \frac{2r_1}{2E\sigma^2 + N_0} \exp\left(-\frac{r_1^2}{K}\right) dr_1$$

where

$$K = \frac{N_0\left(2E\sigma^2 + N_0\right)}{2E\sigma^2 + N_0}$$

The remaining integral is then easily integrated since its integrand is a perfect differential if written as

$$P\left(E|H_1\right) = \frac{K}{2E\sigma^2 + N_0} \int_0^\infty \frac{2r_1}{K} \exp\left(-\frac{r_1^2}{K}\right) dr_1$$

Problem 9.18

a. Decide hypothesis i is true if

$$f_{Z|H_i}\left(z|H_i\right) \geq f_{Z|H_j}(z|H_j), \qquad \text{all } j$$

where

$$f_{Z|H_i}(z|H_i) = \frac{\exp\left(-\frac{z_{ci}^2+z_{si}^2}{E_i\sigma^2+N_0}\right)}{\pi^M (E_i\sigma^2+N_0) N_0^M} \exp\left(-\frac{1}{N_0}\sum_{j=1,j\neq i}^{M}\left(z_{cj}^2+z_{sj}^2\right)\right)$$

This constitutes the maximum likelihood decision rule since the hypotheses are equally probable. Alternatively, we can take the natural log of both sides and use that as a test. This reduces to computing

$$\frac{E_i\sigma^2}{E_i\sigma^2+N_0}\left(z_{ci}^2+z_{si}^2\right)$$

and choosing the signal corresponding to the largest. If the signals have equal energy, then the optimum receiver is seen to be an M-signal replica of the one shown in Fig. 9.8a.

b. The probability of correct decision can be reduced to

$$P_c = \sum_{i=0}^{M-1}\binom{m-1}{i}\frac{N_0}{N_0+(E\sigma^2+N_0)(M-1-i)} = 1-P_e$$

Problem 9.19

a. It can be shown under hypothesis that H_1, Y_1 is the sum of squares of $2N$ independent Gaussian random variables each having mean zero and variance

$$\sigma_{11}^2 = 2\frac{E}{N}\sigma^2+\frac{N_0}{2}$$

and Y_2 is the sum of squares of $2N$ independent Gaussian random variables each having zero mean and variance

$$\sigma_{21}^2 = \frac{N_0}{2}$$

Therefore, from Problem 4.37, the pdfs under hypothesis H_1 are

$$f_{Y_i}(y_1|H_1) = \frac{y_1^{N-1}e^{-y_1/2\sigma_{11}^2}}{2^N\sigma_{11}^N\Gamma(N)}, \qquad y_1 \geq 0$$

and

$$f_{Y_2}(y_2|H_1) = \frac{y_2^{N-1}e^{-y_2/N_0}}{2^N(N_0/2)^N\Gamma(N)}, \qquad y_2 \geq 0$$

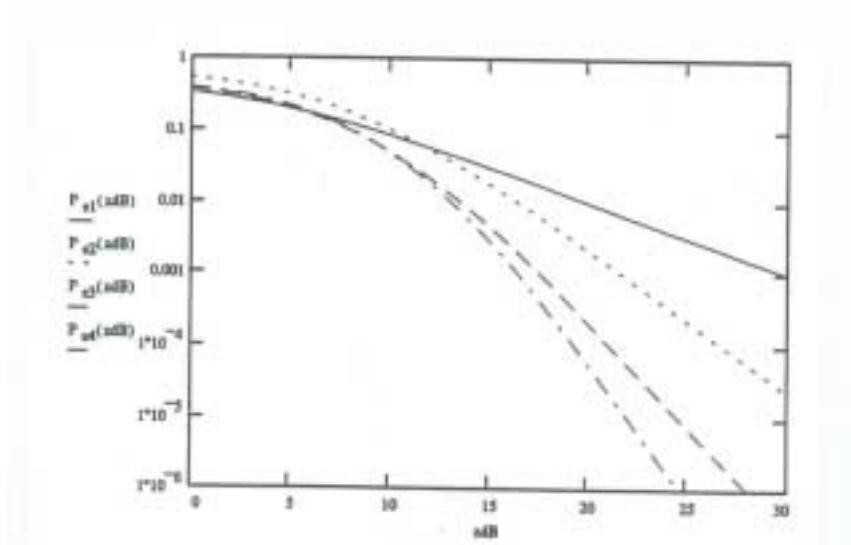

Figure 9.4:

b. The probability of error is

$$\begin{aligned} P_e &= \Pr(E|H_1) = \Pr(E|H_2) = \Pr(Y_2 > Y_1|H_1) \\ &= \int_0^\infty \left[\int_{y_i}^\infty f_{Y_2}(y_2|H_1)\,dy_2\right] f_{Y_1}(y_1|H_1)\,dy_1 \end{aligned}$$

Using integration by parts, it can be shown that

$$I(x;\,a) = \int_x^\infty z^n e^{-az} dz = e^{-ax} \sum_{i=0}^{n} x^i \frac{n!}{i!a^{n+1-i}}$$

This can be used to get the formula given in the problem statement.

c. Plots are given in Figure 9.4 for $N = 1$ (solid curve), 2, 3, and 4 (dash-dot curve).

Problem 9.20

a. The characteristic function is found and used to obtain the moments. By definition,

$$\Phi(j\omega) = E\left[e^{j\Lambda\omega}\right] = \int_0^\infty e^{j\lambda\omega} \frac{\beta^m}{\Gamma(m)} e^{-\beta\lambda} \lambda^{m-1} d\lambda$$

To integrate, note that

$$\int_0^\infty e^{j\lambda\omega} e^{-\beta\lambda} d\lambda = \frac{1}{\beta - j\omega}$$

Differentiate both sides $m-1$ times with respect to β. This gives

$$\int_0^\infty e^{j\lambda\omega} (-1)^{m-1} \lambda^{m-1} e^{-\beta\lambda} d\lambda = \frac{(-1)^{m-1} (m-1)!}{(\beta - j\omega)^m}$$

Multiply both sides by β^m and cancel like terms to get

$$\int_0^\infty e^{j\lambda\omega} \frac{\beta^m}{\Gamma(m)} e^{-\beta\lambda} \lambda^{m-1} d\lambda = \frac{\beta^m}{(\beta - j\omega)^m}$$

Thus

$$\Phi(j\omega) = \frac{\beta^m}{(\beta - j\omega)^m}$$

Differentiating this with respect to ω, we get

$$E[\Lambda] = j\Phi'(0) - \frac{m}{\beta} \qquad \text{and} \qquad E\left[\Lambda^2\right] = j^2\Phi''(0) = \frac{m(m+1)}{\beta}$$

The variance is

$$Var(\Lambda) = E\left(\Lambda^2\right) - E^2(\Lambda) = \frac{m}{\beta^2}$$

b. Use Bayes' rule. But first we need $f_Z(z)$, which is obtained as follows:

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_{Z|\Lambda}(z|\lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^\infty \lambda e^{-\lambda z} \frac{\beta^m}{\Gamma(m)} e^{-\beta\lambda} \lambda^{m-1} d\lambda \\ &= \frac{m\beta^m}{(\beta+z)^{m+1}} \int_0^\infty e^{(\beta+z)\lambda} \frac{(\beta+z)^{m+1}}{\Gamma(m+1)} \lambda^m d\lambda \\ &= \frac{m\beta^m}{(\beta+z)^{m+1}} \end{aligned}$$

where the last integral is evaluated by noting that the integrand is a pdf and therefore integrates to unity. Using Bayes' rule, we find that

$$f_{\Lambda|Z}(\lambda|z) = \frac{\lambda^m (\beta+z)^{m+1} e^{-(\beta+z)\lambda}}{\Gamma(m+1)}$$

Note that this is the same pdf as in part (a) except that β has been replaced by $\beta + z$ and m has been replaced by $m+1$. Therefore, we can infer the moments from part (a) as well. They are

$$E\left[\Lambda|Z\right] = \frac{m+1}{\beta + Z} \qquad \text{and} \qquad Var\left[\Lambda|Z\right] = \frac{m+1}{\left(\beta + Z\right)^2}$$

c. Assume the observations are independent. Then

$$f_{z_1 z_2|\Lambda}\left(z_1, z_2|\lambda\right) = \lambda^2 e^{-\lambda(z_1+z_2)}, z_1, z_2 \geq 0$$

The integration to find the joint pdf of Z_1 and Z_2 is similar to the procedure used to find the pdf of Z above with the result that

$$f_{z_1 z_2}\left(z_1, z_2\right) = \frac{m\left(m+1\right)\beta^m}{\left(\beta + z_1 + z_2\right)^{m+2}}$$

Again using Bayes' rule, we find that

$$f_{\Lambda|z_1 z_2}\left(\lambda|z_1, z_2\right) = \frac{\lambda^{m+1} e^{-\lambda(\beta+z_1+z_2)}\left(\beta + z_1 + z_2\right)^{m+2}}{\Gamma\left(m+2\right)}$$

Since this is of the same form as in parts (a) and (b), we can extend the results there for the moments to

$$E\left[\Lambda|Z_1, Z_2\right] = \frac{m+2}{\beta + Z_1 + Z_2} \text{ and } Var\left[\Lambda|Z_1, Z_2\right] = \frac{m+2}{\left(\beta + Z_1 + Z_2\right)^2}$$

d. By examining the pattern developed in parts. (a), (b), and (c), we conclude that

$$f_{\Lambda|z_1 z_2 \ldots z_K}\left(\lambda|z_1, z_2, \ldots z_K\right) = \frac{\lambda^{m+1+K} e^{-\lambda(\beta+z_1+z_2+\ldots+z_K)}\left(\beta + z_1 + z_2 + \ldots + z_K\right)^{m+K}}{\Gamma\left(m+2\right)}$$

and

$$\begin{aligned} E\left[\Lambda|Z_1, Z_2, \ldots, Z_K\right] &= \frac{m+2}{\beta + Z_1 + Z_2 + \ldots + Z_K} \\ Var\left[\Lambda|Z_1, Z_2, \ldots, Z_K\right] &= \frac{m+K}{\left(\beta + Z_1 + Z_2 + \ldots + Z_K\right)^2} \end{aligned}$$

Problem 9.21

The conditional mean and Bayes' estimates are the same if:

1. $C(x)$ is symmetric;
2. $C(x)$ is convex upward;
3. $f_{A|Z}$ is symmetric about the mean.

With this, we find the following to be true:

a. Same - all conditions are satisfied;

b. Same - all conditions are satisfied;

c. Condition 3 is not satisfied;

d. Condition 2 is not satisfied.

Problem 9.22

a. The conditional pdf of the noise samples is

$$f_Z\left(z_1, z_2, \ldots, z_K | \sigma_n^2\right) = \left(2\pi\sigma_n^2\right)^{-K/2} e^{-\sum_{i=1}^{K} z_i^2/2\sigma_n^2}$$

The maximum likelihood estimate for σ_n^2 maximizes this expression. It can be found by differentiating the pdf with respect to σ_n^2 and setting the result equal to zero. Solving for σ_n^2, we find the maximum likelihood estimate to be

$$\widehat{\sigma}_n^2 = \frac{1}{K}\sum_{i=1}^{K} Z_i^2$$

b. The variance of the estimate is

$$Var\left(\widehat{\sigma}_n^2\right) = \frac{2\sigma_n^4}{K}$$

c. Yes.

d. $\sum_{i=1}^{K} Z_i^2$ is a sufficient statistic.

Problem 9.23

We base our estimate on the observation Z_0, where $Z_0 = M_0 + N$ where N is a Gaussian random variable of mean zero and variance $N_0/2$. The sample value M_0 is Gaussian with mean m_0 and variance σ_m^2. The figure of merit is mean-squared error. Thus, we use a MAP estimate. It can be shown that the estimate for m_0 is the mean value of M_0 conditioned on Z_0 because the random variables involved are Gaussian. That is,

$$\widehat{m}_0 = E\left(m_0 | Z_0 = z_0\right) = \rho z_0$$

where

$$\rho = \frac{E\left(M_0 Z_0\right)}{\sqrt{Var\left(M_0\right)}\sqrt{Var\left(Z_0\right)}} = \frac{\sigma_m}{\sqrt{\sigma_m^2 + \frac{N_0}{2}}}$$

It is of interest to compare this with the ML estimate, which is

$$\widehat{m}_{0,ML} = Z_0$$

which results if we have no *a priori* knowledge of the parameter. Note that the MAP estimate reduces to the ML estimate when $\sigma_m^2 \to \infty$. Note also that for $N_0 \to \infty$ (i.e., small signal-to-noise ratio) we don't use the observed value of Z_0, but instead estimate m_0 as zero (its mean).

Problem 9.24

a. The results are

$$f_Z\left(z_1, z_2 | \theta, H_{1,2}\right) = \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left[(z_1 \mp A)^2 + (z_2 \pm A)^2\right]}$$

where the top sign in the exponent goes with H_1 and the bottom sign goes with H_2,

b. The result is

$$f_{Z|\Theta}\left(Z_1, Z_2 | \Theta\right) = \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left(Z_1^2 + Z_2^2 + \frac{T}{2}A^2\right)} \cosh\left[\frac{2}{N_0}\left(A_1 Z_1 - A_2 Z_2\right)\right]$$

Since the log function is monotonic, we can take the natural log of both sides and differentiate it with respect to Θ, set the result to zero, and have a condition that the maximum likelihood estimate must satisfy. The result is

$$\tanh\left[\sqrt{\frac{T}{2}}\frac{2A}{N_0}\left(Z_1 \cos\theta - Z_2 \sin\theta\right)\right]\sqrt{\frac{T}{2}}\frac{2A}{N_0}\left(-Z_1 \sin\theta - Z_2 \cos\theta\right) = 0$$

Using definitions for Z_1 and Z_2, which are

$$Z_1 = \int_0^T y(t)\sqrt{\frac{2}{T}}\cos\omega_c t\,dt \text{ and } Z_2 = \int_0^T y(t)\sqrt{\frac{2}{T}}\sin\omega_c t\,dt$$

this can be put into the form

$$\tanh\left[\frac{4z}{AT}\int_0^T y(t)\cos(\omega_c t+\theta)\,dt\right]\frac{4z}{AT}\int_0^T -y(t)\sin(\omega_c t+\theta)\,dt = 0$$

where $z = A^2T/2N_0$. This implies a Costas loop type structure with integrators in place of the low pass filters and a tanh-function in the leg with the cosine multiplication. Note that for x small $\tanh(x) \simeq x$ and the phase estimator becomes a Costas loop if the integrators are viewed as lowpass filters.

c. The variance of the phase estimate is bounded by

$$Var\left(\widehat{\theta}_{ML}\right) \geq \frac{1}{4z(z+1)}$$

Problem 9.24

This is similar to the previous problem. For (a) note that

$$\cos\left[\cos^{-1} m\right] = m, \qquad \sin\left[\cos^{-1} m\right] = \left(1-m^2\right)^{1/2}$$

The development of the log-likelihood function follows similarly to the development of the one for Problem 9.23. The block diagram of the phase estimator has another arm that adds into the feedback to the VCO that acts as a phase-lock loop for the carrier component.

Chapter 10

Information Theory and Coding

10.1 Problem Solutions

Problem 10.1
The information in the message is

$$\begin{aligned} I(x) &= -\log_2(0.8) = 0.3219 \text{ bits} \\ I(x) &= -\log_e(0.8) = 0.2231 \text{ nats} \\ I(x) &= -\log_{10}(0.8) = 0.0969 \text{ Hartleys} \end{aligned}$$

Problem 10.2
(a) $I(x) = -\log_2(52) = 5.7004$ bits
(b) $I(x) = -\log_2[(52)(52)] = 11.4009$ bits
(c) $I(x) = -\log_2[(52)(51)] = 11.3729$ bits

Problem 10.3
The entropy is

$$\begin{aligned} I(x) &= -0.3\log_2(0.3) - 0.25\log_2(0.25) \\ &\quad -0.25\log_2(0.25) - 0.1\log_2(0.1) \\ &\quad -0.05\log_2(0.05) - 0.05\log_2(0.05) = 2.1477 \text{ bits} \end{aligned}$$

The maximum entropy is

$$I(x) = \log_2 5 = 2.3219 \text{ bits}$$

Problem 10.4

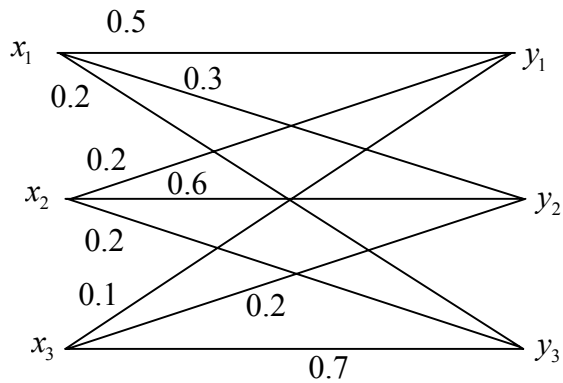

Figure 10.1:

$$I(x) = 2.471 \text{ bits}$$

Problem 10.5

(a) The channel diagram is shown is illustrated in Figure 10.1. (b) The output probabilities are

$$\begin{aligned} p(y_1) &= \frac{1}{3}(0.5 + 0.2 + 0.1) = \frac{8}{30} = 0.2667 \\ p(y_2) &= \frac{1}{3}(0.3 + 0.6 + 0.2) = \frac{11}{30} = 0.3667 \\ p(y3) &= \frac{1}{3}(0.2 + 0.2 + 0.7) = \frac{11}{30} = 0.3667 \end{aligned}$$

(c) Since

$$[P(Y)] = [P(X)][P(Y|X)]$$

we can write

$$[P(X)] = [P(Y)][P(Y|X)]^{-1}$$

which is

$$[P(X)] = [0.333 \quad 0.333 \quad 0.333] \begin{bmatrix} 2.5333 & -1.1333 & -0.4000 \\ -0.8000 & 2.2000 & -0.4000 \\ -0.1333 & -0.4667 & 1.6000 \end{bmatrix}$$

This gives$[P(X)] = [0.5333 \quad 0.2000 \quad 02.667]$
(d) The joint probability matrix is

$$[P(X;Y)] = [0.333 \quad 0.333 \quad 0.333] \begin{bmatrix} 0.5333 & 0 & 0 \\ 0 & 0.2000 & 0 \\ 0 & 0 & 0.2667 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

which gives

$$[P(X;Y)] = \begin{bmatrix} 0.2667 & 0.1600 & 0.1067 \\ 0.0400 & 0.1200 & 0.0400 \\ 0.0267 & 0.0533 & 0.1867 \end{bmatrix}$$

Note that the column sum gives the output possibilities $[P(Y)]$, and the row sum gives the input probabilities $[P(X)]$.

Problem 10.6
For a noiseless channel, the transition probability matrix is a square matrix with 1s on the main diagonal and zeros elsewhere. The joint probability matrix is a square matrix with the input probabilities on the main diagonal and zeros elsewhere. In other words

$$[P(X;Y)] = \begin{bmatrix} p(x_1) & 0 & \cdots & 0 \\ 0 & p(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(x_n) \end{bmatrix}$$

Problem 10.7
This problem may be solved by raising the channel matrix

$$A = \begin{bmatrix} 0.999 & 0.001 \\ 0.001 & 0.999 \end{bmatrix}$$

which corresponds to an error probability of 0.001, to increasing powers n and seeing where the error probability reaches the critical value of 0.08. Consider the MATLAB program

```
a = [0.999 0.001; 0.001 0.999];    % channel matrix
n = 1;                             % initial value
a1 = a;                            % save a
while a1(1,2)<0.08
    n=n+1;
    a1=a^n;
end
```

```
n-1                                    % display result
```

Executing the program yields $n-1=87$. Thus we compute (MATLAB code is given)

```
≫ a^87
ans =
 0.9201 0.0799
 0.0799 0.9201
≫ a^88
ans =
 0.9192 0.0808
 0.0808 0.9192
```

Thus 87 cascaded channels meets the specification for $P_E < 0.08$. However cascading 88 channels yields $P_E > 0.08$ and the specification is not satisfied. (Note: This may appear to be an impractical result since such a large number of cascaded channels are specified. However, the channel A may represent a lengthly cable with a large number of repeaters. There are a number of other practical examples.)

Problem 10.8

The first step is to write $H(Y|X) - H(Y)$. This gives

$$H(Y|X) - H(Y) = -\sum_i \sum_j p(x_i, y_j) \log_2 p(y_j|x_i) - \sum_j p(y_j) \log_2 p(y_j)$$

which is

$$H(Y|X) - H(Y) = -\sum_i \sum_j p(x_i, y_j) \left[\log_2 p(y_j|x_i) - \log_2 p(y_i)\right]$$

or

$$H(Y|X) - H(Y) = -\frac{1}{\ln 2} \sum_i \sum_j p(x_i, y_j) \ln \left[\frac{p(x_i, y_j)}{p(x_i)p(x_i)}\right]$$

or

$$H(Y|X) - H(Y) = \frac{1}{\ln 2} \sum_i \sum_j p(x_i, y_j) \ln \left[\frac{p(x_i)(y_j)}{p(x_i, y_j)}\right]$$

Since $\ln x \leq x - 1$, the preceding expression can be written

$$H(Y|X) - H(Y) \leq \frac{1}{\ln 2} \sum_i \sum_j p(x_i, y_j) \left[\frac{p(x_i)p(y_j)}{p(x_i, y_j)} - 1\right]$$

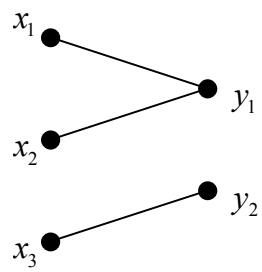

Figure 10.2:

or

$$H(Y|X) - H(Y) \leq \frac{1}{\ln 2}\left\{\sum_i \sum_j p(x_i)p(y_j) - \sum_i \sum_j p(x_i, y_j)\right\}$$

The term in braces is zero since both double sums evaluate to one. Thus,

$$H(Y|X) - H(Y) \leq 0$$

which gives

$$H(Y) \geq H(Y|X)$$

Problem 10.9

For this problem note that $H(Y|X) = 0$ so that $I(Y|X) = H(X)$. It therefore follows that

$$C = 1 \text{ bit/symbol}$$

and that the capacity is achieved for

$$p(x_1) = p(x_2) = \frac{1}{2}$$

Problem 10.10

For this problem, the channel diagram is illustrated in Figure 10.2. Note that

$$\begin{aligned} p(y_1|x_1) &= 1 \qquad p(y_2|x_1) = 0 \\ p(y_1|x_2) &= 1 \qquad p(y_2|x_2) = 0 \\ p(y_1|x_3) &= 0 \qquad p(y_2|x_3) = 1 \end{aligned}$$

Thus

$$H(Y|X) = -\sum_i \sum_j p(x_i, y_j) \log_2 p(y_j|x_i) = 0$$

since each term in the summand is zero. Thus

$$I(X;Y) = H(Y)$$

The capacity is therefore

$$C = 1 \qquad \text{bit/symbol}$$

and is achieved for source probabilities satisfying

$$p(x_1) + p(x_2) = \frac{1}{2} \qquad p(x_3) = \frac{1}{2}$$

Note that capacity is not achieved for equally likely inputs.

Problem 10.11
For the erasure channel

$$C = p \qquad \text{bits/symbol}$$

and capacity is achieved for equally likely inputs.

Problem 10.12
The capacity of the channel described by the transition probability matrix

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

is easily computed by maximizing

$$I(X;Y) = H(Y) - H(Y|X)$$

The conditional entropy $H(Y|X)$ can be written

$$H(Y|X) = -\sum_i \sum_j p(x_i) p(y_j|x_i) \log_2 p(y_j|x_i) = \sum_i p(x_i) H(Y|x_i)$$

where

$$H(Y|x_i) = -\sum_j p(y_j|x_i) \log_2 p(y_j|x_i)$$

For the given channel

$$H(Y|x_i) = -p\log_2 p - q\log_2 q = K$$

so that $H(Y|x_i)$ is a constant independent of i. This results since each row of the matrix contains the same set of probabilities although the terms are not in the same order. For such a channel

$$H(Y|X) = \sum_i p(x_i) K = K$$

and

$$I(X;Y) = H(Y) - K$$

We see that capacity is achieved when each channel output occurs with equal probability.

We now show that equally likely outputs result if the inputs are equally likely. Assume that $p(x_i) = \frac{1}{4}$ for all i. Then

$$p(y_j) = \sum_i p(x_i) p(y_j|x_i)$$

For each j

$$p(y_j) = \frac{1}{4}\sum_i p(y_j|x_i) = \frac{1}{4}(p+q) = \frac{1}{4}[p+(1-p)] = \frac{1}{4}$$

so that

$$p(y_j) = \frac{1}{4}, \qquad \text{all } j$$

Since $p(y_j) = \frac{1}{4}$ for all j, $H(Y) = 2$, and the channel capacity is

$$C = 2 + p\log_2 p + (1-p)\log_2(1-p)$$

or

$$C = 2 - H(p)$$

This is shown in Figure 10.3.

This channel is modeled as a pair of binary symmetric channels as shown. If $p = 1$ or $p = 0$, each input uniquely determines the output and the channel reduces to a noiseless channel with four inputs and four outputs. This gives a capacity of $\log_2 4 = 2$ bits/symbol. If $p = q = \frac{1}{2}$, then each of the two subchannels (Channel 1 and Channel 2) have zero capacity. However, x_1 and x_2 can be viewed as a single input and x_3 and x_4 can be viewed as a single input as shown in Figure 10.4. The result is equivalent to a noiseless channel with two inputs and two outputs. This gives a capacity of $\log_2 2 = 1$ bit/symbol.

Note that this channel is an example of a general symmetric channel. The capacity of such channels are easily computed. See Gallager (1968), pages 91-94 for a discussion of these channels.

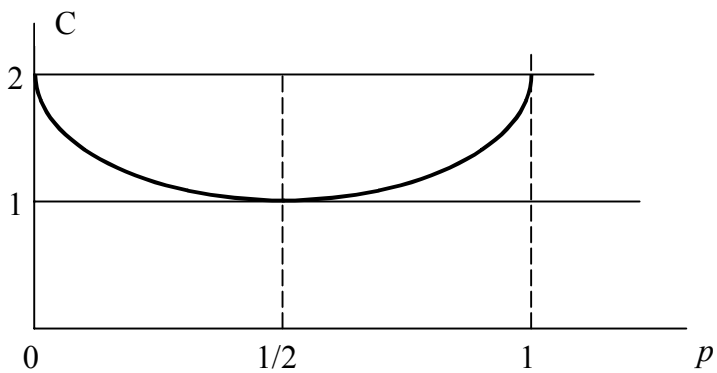

Figure 10.3:

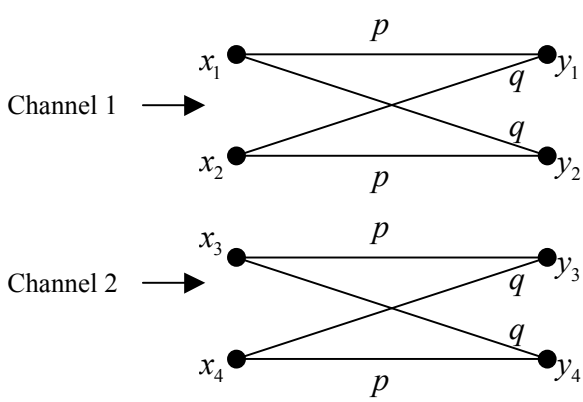

Figure 10.4:

Problem 10.13
To show (10.30), write $p(x_i, y_j)$ in (10.28) as $p(x_i|y_j)\,p(y_j)$ and recall that

$$\log_2 p(x_i, y_j) = \log_2 p(x_i|y_j) + \log_2 p(y_j)$$

Evaluating the resulting sum yields (10.30). The same method is used to show (10.31) except that $p(x_i, y_j)$ is written $p(y_j|x_i)\,p(x_i)$.

Problem 10.14
The entropy is maximized when each quantizing region occurs with probability 0.25. Thus

$$\int_0^{x_1} ae^{-ax}dx = 1 - e^{ax_1} = \frac{1}{4}$$

which gives

$$e^{-ax_1} = \frac{3}{4}$$

or

$$x_1 = \frac{1}{a}\ln\left(\frac{4}{3}\right) = 0.2877/a$$

In a similar manner

$$\int_{x_1}^{x_2} ae^{-ax}dx = e^{-ax_1} - e^{-ax_2} = \frac{3}{4} - e^{-ax_2} = \frac{1}{4}$$

which gives

$$e^{-ax_2} = \frac{1}{2}$$

or

$$x_2 = \frac{1}{a}\ln(2) = 0.6931/a$$

Also

$$\int_{x_2}^{x_3} ae^{-ax}dx = \frac{1}{4}$$

gives

$$x_3 = 1.3863/a$$

Problem 10.15
The thresholds are now parameterized in terms of σ. The results are

$$\begin{aligned} x_1 &= 0.7585\sigma \\ x_2 &= 1.1774\sigma \\ x_3 &= 1.6651\sigma \end{aligned}$$

Problem 10.16

First we compute the probabilities of the six messages. Using symmetry, we only need compute three states; m_3, m_4, and m_5.

$$\begin{aligned} p(m_3) &= \frac{1}{\sqrt{2\pi}\sigma}\int_0^{\sigma} e^{-y^2/2\sigma^2}dy = \frac{1}{2} - Q(1) = 0.3413 \\ p(m_4) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{\sigma}^{2\sigma} e^{-y^2/2\sigma^2}dy = Q(1) - Q(2) = 0.1359 \\ p(m_5) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{2\sigma}^{\infty} e^{-y^2/2\sigma^2}dy = Q(2) = 0.0228 \end{aligned}$$

By symmetry

$$\begin{aligned} p(m_0) &= p(m_5) = 0.0228 \\ p(m_1) &= p(m_4) = 0.1359 \\ p(m_2) &= p(m_3) = 0.3413 \end{aligned}$$

the entropy at the quantizer output is

$$\begin{aligned} H(X) &= -2\left[0.3413\ \log_2\ 0.3413 + 0.1359\ \log_2\ 0.1359 + 0.0228\ \log_2\ 0.0228\right] \\ &= 2.09 \text{ bits/symbol} \end{aligned}$$

Since the symbol (sample) rate is 500 samples/s, the information rate is

$$r = 500\ H(X) = 1045 \text{ symbols/s}$$

Problem 10.17

Five thresholds are needed to define the six quantization regions. The five thresholds are $-k_2$, $-k_1$, 0, k_1, and k_2. The thresholds are selected so that the six quantization regions are equally likely. Finding the inverse Q-function yields

$$k_1 = 0.4303 \qquad k_2 = 0.9674$$

This gives the quantizing rule

Quantizer Input	Quantizer Output
$-\infty < x_i < -0.9674\sigma$	m_0
$-0.9674\sigma < x_i < -0.4303\sigma$	m_1
$-0.4303\sigma < x_i < 0$	m_2
$0 < x_i < 0.4303\sigma$	m_3
$0.4303\sigma < x_i < 0.9674\sigma$	m_4
$0.9674\sigma < x_i < \infty$	m_5

Problem 10.18
The transition probabilities for the cascade of Channel 1 and Channel 2 are

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Therefore, the capacity of the overall channel is

$$C = 1 + 0.7 \ \log_2 \ 0.7 + 0.3 \ \log_2 \ 0.3 = 0.1187 \text{ bits/symbol}$$

The capacity of Channel 1 is

$$C = 1 + 0.9 \ \log_2 \ 0.9 + 0.1 \ \log_2 \ 0.1 = 0.5310 \text{ bits/symbol}$$

and the capacity of Channel 2 is

$$C = 1 + 0.75 \ \log_2 \ 0.75 + 0.25 \ \log_2 \ 0.25 = 0.1887 \text{ bits/symbol}$$

It can be seen that the capacity of the cascade channel is less than the capacity of either channel taken alone.

Problem 10.19
For BPSK, the error probability is $Q\left(\sqrt{2z}\right)$. From the data given

$$\begin{aligned} \text{Uplink error probability} &= 0.000191 \\ \text{Downlink error probability} &= 0.005954 \end{aligned}$$

Using the techniques of the previous problem, or equivalently, Equation (10.24), yields the overall error probability

$$P_E = 0.006143$$

Note that for all practical purposes the performance of the overall system is established by the downlink performance.

Problem 10.20
The source entropy is

$$H(X) = -\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 0.8113$$

and the entropy of the fourth-order extension is

$$H\left(X^4\right) = 4H(X) = 4(0.3113) = 3.2451$$

The second way to determine the entropy of the fourth-order extension, is to determine the probability distribution of the extended source. We have

$$\begin{array}{l} 1 \text{ symbol with probability } \dfrac{81}{256} \\ 4 \text{ symbols with probability } \dfrac{27}{256} \\ 6 \text{ symbols with probability } \dfrac{9}{256} \\ 4 \text{ symbols with probability } \dfrac{3}{256} \\ 1 \text{ symbol with probability } \dfrac{1}{256} \end{array}$$

Thus

$$\begin{aligned} H\left(X^4\right) &= -\frac{81}{256}\log_2\frac{81}{256} - 4\left(\frac{27}{256}\log_2\frac{27}{256}\right) - 6\left(\frac{9}{256}\log_2\frac{9}{256}\right) \\ &\quad -4\left(\frac{3}{256}\log_2\frac{3}{256}\right) - \frac{1}{256}\log_2\frac{1}{256} \\ &= 3.2451 \text{ bits/symbol} \end{aligned}$$

Problem 10.21

For the fourth-order extension we have

Source Symbol	**P** $(\cdot)$	**Codeword**
AAAA	0.6561	0
BAAA	0.0729	100
ABAA	0.0729	101
AABA	0.0729	110
AAAB	0.0729	1110
AABB	0.0081	111100
ABAB	0.0081	1111010
BAAB	0.0081	1111011
ABBA	0.0081	1111100
BABA	0.0081	1111101
BBAA	0.0081	1111110
ABBB	0.0009	111111100
BABB	0.0009	111111101
BBAB	0.0009	111111110
BBBA	0.0009	1111111110
BBBB	0.0001	1111111111

The average wordlength: $\overline{L} = 1.9702$ which is $\overline{L}/4 = 1.9702/4 = 0.4926$. The efficiencies are given in the following table:

n	$\overline{L}$	$H(X^n)$	**Efficiency**
1	1	0.4690	46.90%
2	1.29	0.9380	72.71%
3	1.598	1.4070	88.05%
4	1.9702	1.8760	95.22%

Problem 10.22

For the Shannon-Fano code we have

Source Symbol	**P** $(\cdot)$	**Codeword**
m_1	0.2	00
m_2	0.2	01
m_3	0.2	10
m_4	0.2	110
m_5	0.2	111

The entropy is

$$H(X) = -5(0.2 \log_2 0.2) = 2.3219$$

The average wordlength is

$$I = (0.2)(2+2+2+3+3) = 2.4$$

The efficiency is therefore given by

$$\frac{H(X)}{\overline{L}} = \frac{2.319}{2.4} = 0.9675 = 96.75\%$$

The diagram for determining the Huffman code is illustrated in Figure 10.5. This diagram yields the codewords

m_1	01
m_2	10
m_3	11
m_4	000
m_5	001

The average wordlength is

$$\overline{L} = 0.2(2+2+2+3+3) = 2.4$$

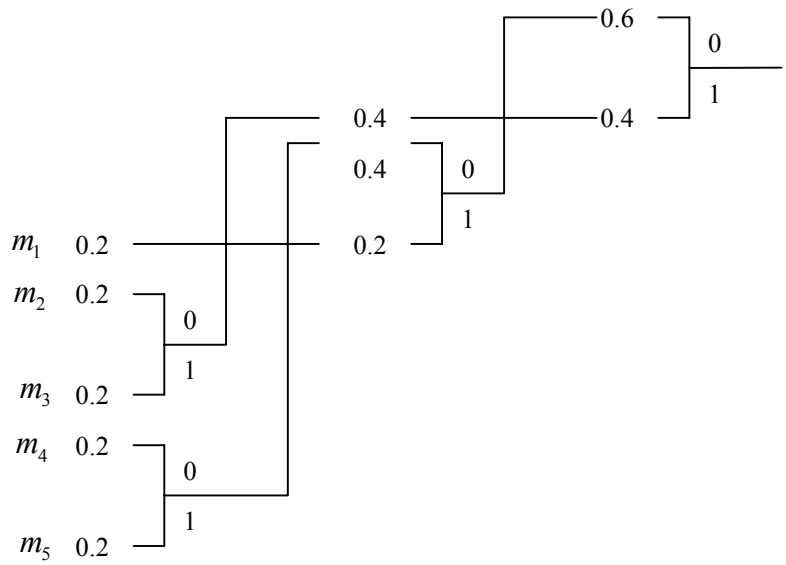

Figure 10.5:

Since the average wordlength is the same as for the Shannon-Fano code, the efficiency is also the same.

Problem 10.23
The codewords for the Shannon-Fano and Huffman codes are summarized in the following table:

		Codewords	
Source Symbol	**P (·)**	**Shannon-Fano**	**Huffman**
m_1	0.40	00	1
m_2	0.19	01	000
m_3	0.16	10	001
m_4	0.15	110	010
m_5	0.10	111	011

The average wordlengths are:

$$\begin{aligned} \overline{L} &= 2.25 \qquad \text{(Shannon-Fano code)} \\ \overline{L} &= 2.22 \qquad \text{(Huffman code)} \end{aligned}$$

Note that the Huffman code gives the shorter average wordlength and therefore the higher efficiency.

Problem 10.24
The entropy is

$$\begin{aligned} H(X) &= -0.85 \ \log_2 \ 0.85 - 0.15 \ \log_2 \ 0.15 \\ &= 0.60984 \text{ bits/symbol} \end{aligned}$$

We know that

$$\lim_{n\to\infty} r\frac{\overline{L}}{n} \leq 300$$

and that

$$\lim_{n\to\infty} \frac{\overline{L}}{n} = H(X)$$

Therefore, for n large

$$rH(X) \leq 300$$

or

$$r \leq \frac{300}{H(X)} = \frac{300}{0.60984}$$

which gives

$$r \leq 491.932 \text{ symbols/s}$$

Problem 10.25
The Shannon-Fano and Huffman codes are as follows. (Note that the codes are not unique.)

	Codewords	
Source Symbol	**Shannon-Fano**	**Huffman**
m_1	000	001
m_2	001	010
m_3	010	011
m_4	011	100
m_5	100	101
m_6	101	110
m_7	110	111
m_8	1110	0000
m_9	1111	0001

In both cases, there are seven codewords of length three and two codewords of length four. Thus, both codes have the same average wordlengths and therefore have the same efficiencies. The average wordlength is

$$\overline{L} = \frac{1}{9}\left[7(3) + 2(4)\right] = \frac{29}{9} = 3.2222$$

Since the source entropy is

$$H(X) = \log_2 9 = 3.1699$$

the efficiency is

$$E = \frac{H(X)}{\overline{L}} = 98.38\%$$

Problem 10.26

For the Shannon-Fano code we have

Source Symbol	**P** $(\cdot)$	**Codeword**
m_1	1/12	000
m_2	1/12	0010
m_3	1/12	0011
m_4	1/12	010
m_5	1/12	0110
m_6	1/12	0111
m_7	1/12	100
m_8	1/12	1010
m_9	1/12	1011
m_{10}	1/12	110
m_{11}	1/12	1110
m_{12}	1/12	1111

The average wordlength is

$$\overline{L} = \frac{1}{12}[4(3) + 8(4)] = \frac{44}{12} = 3.6667$$

and the entropy of the source is

$$H(X) = \log_2 12 = 3.5850$$

This gives the efficiency

$$E = \frac{H(X)}{\overline{L}} = 0.9777 = 97.77\%$$

The diagram for determining the Huffman code is illustrated in Figure 10.6. This yields

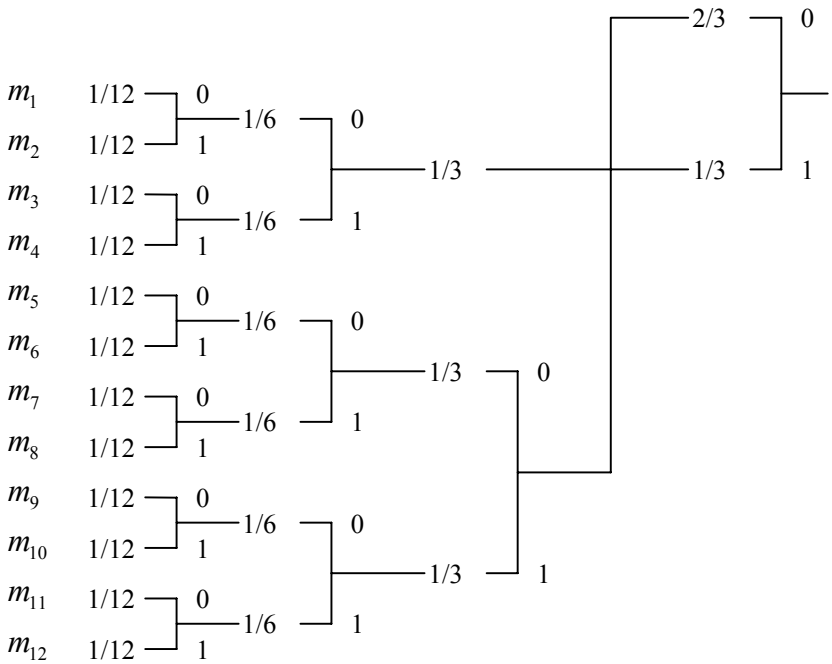

Figure 10.6:

the codewords:

Source Symbol	**Codeword**
m_1	100
m_2	101
m_3	110
m_4	111
m_5	0000
m_6	0001
m_7	0010
m_8	0011
m_9	0100
m_{10}	0101
m_{11}	0110
m_{12}	0111

As in the case of the Shannon-Fano code, we have four codewords of length three and eight codewords of length four. The average wordlength is therefore 3.6667 and the efficiency is

97.77%.

Problem 10.27
The probability of message m_k is

$$P(m_k) = \int_{0.1(k-1)}^{0.1k} 2x\,dx = 0.01\,(2k-1)$$

A Huffman code yields the codewords in the following table:

Source Symbol	**P (·)**	**Codeword**
m_{10}	0.19	11
m_9	0.17	000
m_8	0.15	010
m_7	0.13	011
m_6	0.11	100
m_5	0.09	101
m_4	0.07	0011
m_3	0.05	00100
m_2	0.03	001010
m_1	0.01	001011

The source entropy is given by

$$H(X) = 3.0488 \text{ bits/source symbol}$$

and the average wordlength is

$$\overline{L} = 3.1000 \text{ binary symbols/source symbol}$$

A source symbol rate of 250 symbols (samples) per second yields a binary symbol rate of

$$r\overline{L} = 250(3.1000) = 775 \text{ binary symbols/second}$$

and a source information rate of

$$R_s = rH(X) = 250(3.0488) = 75.2 \text{ bits/second}$$

Problem 10.28
The source entropy is

$$\begin{aligned} H(X) &= -0.4\ \log_2\ 0.4 - 0.3\ \log_2\ 0.3 - 0.2\ \log_2\ 0.2 - 0.1\ \log_2\ 0.1 \\ &= 1.84644 \text{ bits/source symbol} \end{aligned}$$

and the entropy of the second-order extension of the source is

$$H\left(X^2\right) = 2H\left(X\right) = 3.69288$$

The second-order extension source symbols are shown in the following table. For a $D = 3$ alphabet we partition into sets of 3. The set of code symbols is $(0, 1, 2)$.

Source Symbol	**P** $(\cdot)$	**Codeword**
m_1m_1	0.16	00
m_1m_2	0.12	01
m_2m_1	0.12	02
m_2m_2	0.09	10
m_1m_3	0.08	11
m_3m_1	0.08	12
m_2m_3	0.06	200
m_3m_2	0.06	201
m_3m_3	0.04	202
m_1m_4	0.04	210
m_4m_1	0.04	211
m_2m_4	0.03	212
m_4m_2	0.03	220
m_3m_4	0.02	221
m_4m_3	0.02	2220
m_4m_4	0.01	2221

The average wordlength of the extended source is

$$\begin{aligned}\overline{L} &= 2\left(0.16 + 0.12 + 0.12 + 0.09 + 0.08 + 0.08\right) \\ &\quad +3\left(0.06 + 0.06 + 0.04 + 0.04 + 0.03 + 0.03 + 0.02\right) \\ &\quad +4\left(0.02 + 0.01\right) \\ &= 2.38\end{aligned}$$

Thus the efficiency is

$$E = \frac{H\left(X^2\right)}{\overline{L}\ \log_2\ D} = \frac{3.69288}{(2.38)\ \log_2\ 3} = 97.9\%$$

Problem 10.29

The set of wordlengths from Table 10.3 is $\{1, 3, 3, 3, 5, 5, 5, 5\}$. The Kraft inequality gives

$$\sum_{i=1}^{8} 2^{l_i} = 2^{-1} + 3\left(2^{-3}\right) + 4\left(2^{-5}\right) = \frac{1}{2} + \frac{3}{8} + \frac{4}{32} = 1$$

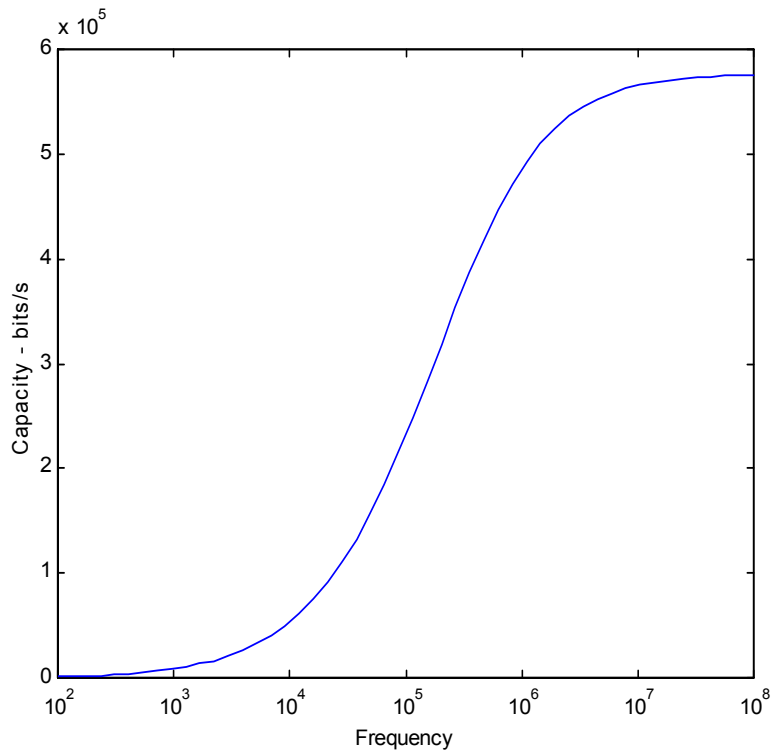

Figure 10.7:

Thus the Kraft inequality is satisfied.

Problem 10.30
From (10.82), the Shannon-Hartley law with $S = 40$ and $N = 10^{-4}B$ gives

$$C_c = B \ \log_2 \ \left(1 + \frac{S}{N}\right) = B \ \log_2 \ \left(1 + \frac{40\left(10^4\right)}{B}\right)$$

This is illustrated in Figure 10.7. It can be seen that as $B \to \infty$ the capacity approaches a finite constant.

Problem 10.31
From the previous problem

$$C_c = B \ \log_2 \ \left(1 + \frac{40\left(10^4\right)}{B}\right)$$

Since $\ln(1+u) \approx u$ for $\ll 1$, we can write that, as $B \to \infty$,

$$C_c = \frac{40\left(10^4\right)}{\ln(2)} = 5.7708\left(10^5\right) \text{ bits/s}$$

Problem 10.32

For a rate $\frac{1}{5}$ repetition code we have, assuming an error probability of $(1-p)$ on a single transmission,

$$P_E = \binom{5}{3} p^2 (1-p)^3 + \binom{5}{4} p (1-p)^4 + \binom{5}{5} (1-p)^5$$

which is

$$P_E = 10p^2 (1-p)^3 + 5p (1-p)^4 + (1-p)^5$$

or

$$P_E = 10 (1-q)^2 q^3 + 5 (1-q) q^4 + q^5$$

where $q = 1-p$. For a rate $\frac{1}{3}$ code we have the corresopnding result (See (10.94) with $q = 1-p$)

$$P_E = 3q^2(1-q) + q^3$$

The results, generated by the MATLAB code,

```
q = 0:0.01:0.1;
p5 = 10*((1-q).^2).*(q.^3)+5*(1-q).*(q.^4)+(q.^5);
p3 = 3*(1-q).*(q.^2)+(q.^3);
plot(q,p3,q,p5)
xlabel('Error probability - (1-p)')
ylabel('Word error probability')
```

are illustrated in Figure 10.8. The top curve (poorest performance) is for the rate $\frac{1}{3}$ code and the bottom curve is for the rate $\frac{1}{5}$ repetition code. Note that these results are not typically compariable since the values of p and q change as the code rate changes.

Problem 10.33

The parity-check matrix for a $(15, 11)$ Hamming code is

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

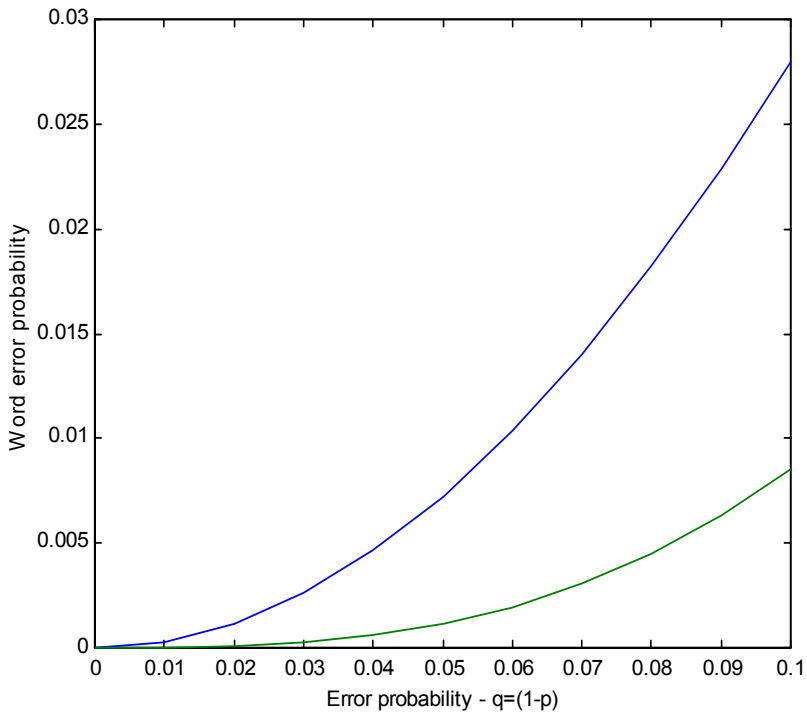

Figure 10.8:

Since the columns of $[H]$ are distinct and all nonzero vectors of length three are included in $[H]$, all single errors can be corrected. Thus,

$$e = \frac{1}{2}(d_m - 1) = 1$$

or

$$d_m = 3$$

Note that it is not necessary to find the set of codewords.

Problem 10.34

The parity-check matrix will have $15 - 11 = 4$ rows and 15 columns. Since rows may be

permuted, the parity-check matrix is not unique. One possible selection is

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding generator matrix is

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{11} \\ H_{11} \end{bmatrix}$$

where I_{11} is the 11×11 identity matrix and H_{11} represents the first 11 columns of the parity-check matrix. From the structure of the parity-check matrix, we see that each parity symbol is the sum of 7 information symbols. Since 7 is odd, the all-ones information sequence yields the parity sequence 1111. This gives the codeword

$$[T]^t = [111111111111111]$$

A single error in the third position yields a syndrome which is the third column of the parity-check matrix. Thus, for the assumed parity-check matrix, the syndrome is

$$[S] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Problem 10.35

For the given parity-check matrix, the generator matrix is

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

This gives the 16 codewords (read vertically)

$\mathbf{a}_1$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$\mathbf{a}_2$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$\mathbf{a}_3$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$\mathbf{a}_4$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$\mathbf{c}_1$	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
$\mathbf{c}_2$	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
$\mathbf{c}_3$	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1

Problem 10.36

We first determine $[T_i] = [G]\,[A_i]$ for $i = 1, 2$. The generator matrix for the previous problem is used. For $[A_1]$ we have

$$[G]\,[A_1] = [G] \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [T_1]$$

and for $[A_2]$ we have

$$[G]\,[A_2] = [G] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = [T_2]$$

We note that

$$[A_1] \oplus [A_2] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$[G]\{[A_1] \oplus [A_2]\} = [G] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which is $[T_1] \oplus [T_2]$.

Problem 10.37
For a rate $\frac{1}{3}$ repetition code

$$[G] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and for a rate $\frac{1}{5}$ repetition code

$$[G] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The generator matrix for a rate $\frac{1}{n}$ repetition code has one column and n rows. All elements are unity.

Problem 10.38
For a Hamming code with $r = n - k$ parity check symbols, the wordlength n is

$$n = 2^r - 1$$

and

$$k = 2^r - r - 1$$

The rate is

$$R = \frac{k}{n} = \frac{2^r - r - 1}{2^r - 1}$$

If n is large, r is large. As $r \to \infty$ both k and n approach 2^r. Thus

$$\lim_{n\to\infty} R = \lim_{r\to\infty} R = \frac{2^r}{2^r} = 1$$

Problem 10.39
The codeword is of the form $(a_1a_2a_3a_4c_1c_2c_3)$ where

$$\begin{aligned} c_1 &= a_1 \oplus a_2 \oplus a_3 \\ c_2 &= a_2 \oplus a_3 \oplus a_4 \\ c_3 &= a_1 \oplus a_2 \oplus a_4 \end{aligned}$$

This gives the generator matrix

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Using the generator matrix, we can determine the codewords. The codewords are (read vertically)

$\mathbf{a}_1$	0	1	1	1	1	0	0	0	1	1	1	0	0	0	0	1
$\mathbf{a}_2$	0	0	1	1	1	0	0	1	0	1	0	1	1	0	1	0
$\mathbf{a}_3$	0	0	0	1	1	0	1	1	1	0	0	1	0	1	0	1
$\mathbf{a}_4$	0	0	0	0	1	1	1	1	1	1	1	0	0	0	1	0
$\mathbf{c}_1$	0	1	0	1	1	0	1	0	0	0	1	0	1	1	1	0
$\mathbf{c}_2$	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1
$\mathbf{c}_3$	0	1	0	0	1	1	1	0	0	1	0	1	1	0	0	1

Let α_i denote the i-th code word and let $\alpha_i \to \alpha_j$ indicate that a cyclic shift of α_i yields α_j. The cyclic shifts are illustrated in Figure 10.9.

Problem 10.40
The parity-check matrix for the encoder in the previous problem is

$$[H] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

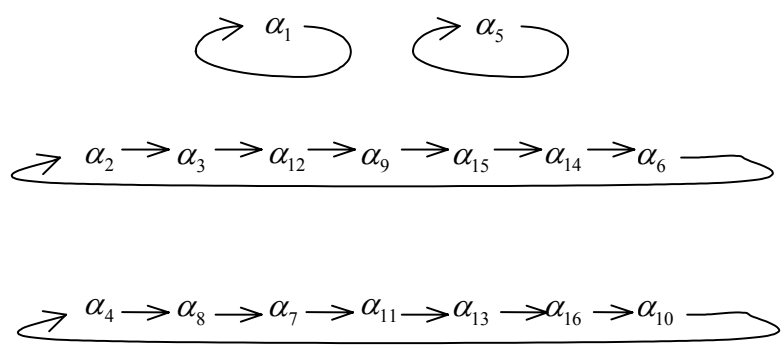

Figure 10.9:

For the received sequence

$$[R]^t = [1101001]$$

the syndrome is

$$[S] = [H]\,[R] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This indicates no errors, which is in agreement with Figure 10.17. For the received sequence

$$[R]^t = [1101011]$$

the syndrome is

$$[S] = [H]\,[R] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This syndrome indicates an error in the 6th position. Thus the assumed transmitted codeword is

$$[T]^t = [1101001]$$

which is also in agreement with Figure 10.17.

Problem 10.41

For the modified encoder we can write

Step	$\mathbf{S}_1$	$\mathbf{S}_2$	$\mathbf{S}_3$
1	a_1	0	0
2	$a_1 \oplus a_2$	a_1	0
3	$a_1 \oplus a_2 \oplus a_3$	$a_1 \oplus a_2$	a_1
4	$a_2 \oplus a_3 \oplus a_4$	$a_1 \oplus a_2 \oplus a_3$	$a_1 \oplus a_2$
5	0	$a_2 \oplus a_3 \oplus a_4$	$a_1 \oplus a_2 \oplus a_3$
6	0	0	$a_2 \oplus a_3 \oplus a_4$
7	0	0	0

Thus, the output for each step is

Step	**Output**
1	a_1
2	a_2
3	a_3
4	a_4
5	$a_1 \oplus a_3 \oplus a_4$
6	$a_1 \oplus a_2 \oplus a_3$
7	$a_2 \oplus a_3 \oplus a_4$

This gives the generator matrix

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Using the generator matrix, the codewords can be constructed. The 16 codewords are (read vertically)

$\mathbf{a}_1$	0	1	1	1	1	0	0	0	1	1	1	0	0	0	0	1
$\mathbf{a}_2$	0	0	1	1	1	0	0	1	0	1	0	1	1	0	1	0
$\mathbf{a}_3$	0	0	0	1	1	0	1	1	1	0	0	1	0	1	0	1
$\mathbf{a}_4$	0	0	0	0	1	1	1	1	1	1	1	0	0	0	1	0
$\mathbf{c}_1$	0	1	1	0	1	1	0	0	1	0	0	1	0	1	1	0
$\mathbf{c}_2$	0	1	0	1	1	0	1	0	0	0	1	0	1	1	1	0
$\mathbf{c}_3$	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1

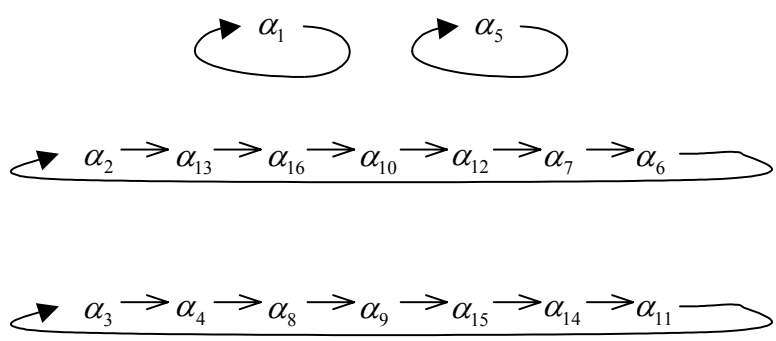

Figure 10.10:

As in Problem 10.39, let α_i denote the i-th code word and let $\alpha_i \to \alpha_j$ indicate that a cyclic shift of α_i yields α_j. These cyclic shifts are illustrated in Figure 10.10.

Problem 10.42

For a $(15, 11)$ Hamming code, the uncoded symbol error probability is given by

$$q_u = Q\left[\sqrt{2z/k}\right]$$

where

$$z = ST_w/N_0$$

and $k = 11$. This gives the uncoded word error probability

$$P_{eu} = 1 - (1 - q_u)^{11}$$

For the coded case, the symbol error probability is

$$q_c = Q\left[\sqrt{2z/15}\right]$$

and the coded word error probability is

$$P_{ec} = \sum_{i=2}^{15} \binom{15}{i} q_c^i (1 - q_c)^{15-i}$$

The performance curves are generated using the MATLAB M-files given in Computer Exercise 10.2 in Section 10.2 (Computer Exercises) of this solutions manual. The results are shown in Figure 10.11. The solid curves represent the symbol error probability with the

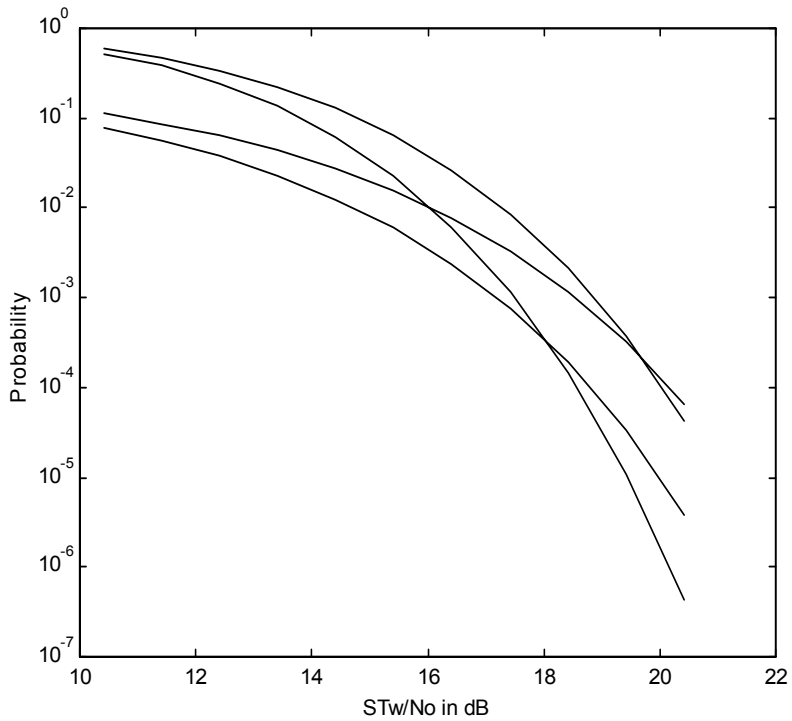

Figure 10.11:

coded case on top (worse performance) and the uncoded case on bottom. The dashed curves represent the word error probability with the coded case (better performance) on bottom.

Problem 10.43

The probability of two errors in a 7 symbol codeword is

$$P\{2 \text{ errors}\} = \binom{7}{2} (1-q_c)^5 q_c^2 = 21 q_c^2 (1-q_c)^5$$

and the probability of three errors in a 7 symbol codeword is

$$P\{3 \text{ errors}\} = \binom{7}{3} (1-q_c)^4 q_c^3 = 35 q_c^3 (1-q_c)^4$$

The ratio of $P\{3 \text{ errors}\}$ to $P\{2 \text{ errors}\}$ is

$$R = \frac{\Pr\{3 \text{ errors}\}}{\Pr\{2 \text{ errors}\}} = \frac{35 q_c^3 (1-q_c)^4}{21 q_c^2 (1-q_c)^5} = \frac{5}{3} \frac{q_c}{1-q_c}$$

We now determine the coded error probability q_c. Since BPSK modulation is used

$$q_c = Q\left(\sqrt{\frac{2z}{7}}\right)$$

where z is ST_w/N_0. This gives us the following table

z	q_c	$R = \frac{\Pr\{3 \text{ errors}\}}{\Pr\{2 \text{ errors}\}}$
$8dB$	0.0897	0.1642
$9dB$	0.0660	0.1177
$10dB$	0.0455	0.0794
$11dB$	0.0290	0.0497
$12dB$	0.0167	0.0278
$13dB$	0.0085	0.0141
$14dB$	0.0037	0.0062

It is clear that as the SNR, z, increases, the value of R becomes negligible.

Problem 10.44

The parity-check matrix for a $(7, 4)$ Hamming code is

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The parity checks are in positions 1,2, and 4. Moving the parity checks to positions 5, 6, and 7 yields a systematic code with the generator matrix

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

which has the partity-check matrix

$$[H] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Note that all columns of $[H]$ are still distinct and therefore all single errors are corrected. Thus, the distance three property is preserved. Since the first four columns of $[H]$ can be in any order, there are $4! = 24$ different $(7, 4)$ systematic codes. If the code is not required to be systematic, there are $7! = 5040$ different codes, all of which will have equivalent performance in an AWGN channel.

Problem 10.45

For the encoder shown, the constraint span is $k = 4$. We first need to compute the states for the encoder. The output is $v_1 v_2$ where $v_1 = s_1 \oplus s_2 \oplus s_3 \oplus s_4$ and $v_2 = s_1 \oplus s_2 \oplus s_4$.

We now compute the state transitions and the output using the state diagram shown in Figure 10.12. This gives the following table.

	Previous			**Current**	
State	$s_1 s_2 s_2$	**Input**	$s_1 s_2 s_3 s_4$	**State**	**Output**
A	000	0	0000	A	00
		1	1000	E	11
B	001	0	0001	A	11
		1	1001	E	00
C	010	0	0010	B	10
		1	1010	F	01
D	011	0	0011	B	01
		1	1011	F	10
E	100	0	0100	C	11
		1	1100	G	00
F	101	0	0101	C	00
		1	1101	G	11
G	110	0	0110	D	01
		1	1110	H	10
H	111	0	0111	D	10
		1	1111	H	01

Problem 10.46

For the encoder shown, we have $v_1 = s_1 \oplus s_2 \oplus s_3$ and $v_2 = s_1 \oplus s_2$. The trellis diagram is illustrated in Figure 10.13. The state transitions, and the output corresponding to each transition appears in the following table.

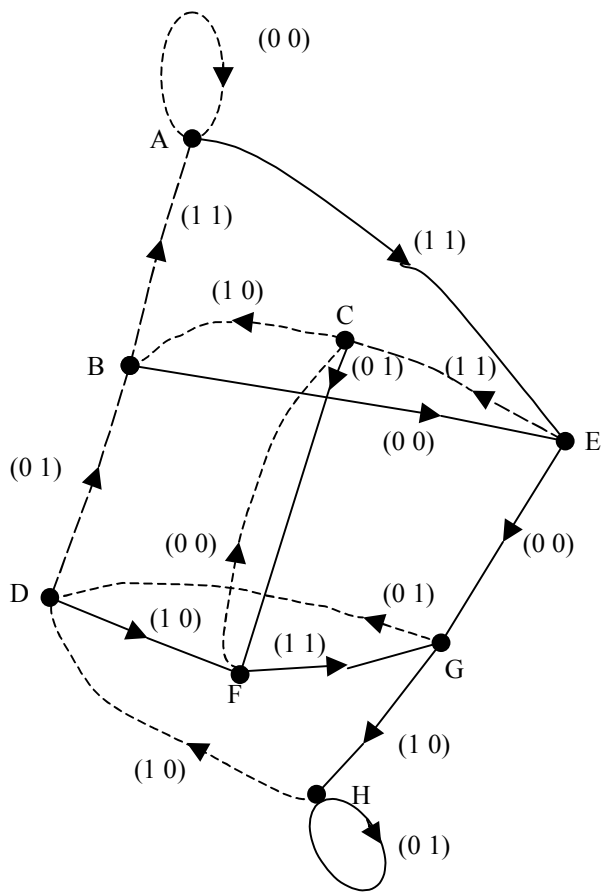

Figure 10.12:

State	s_1s_2	**Input**	$s_1s_2s_3$	**State**	**Output**
A	00	0	000	A	00
		1	100	C	11
B	01	0	001	A	11
		1	101	C	00
C	10	0	010	B	10
		1	110	D	01
D	11	0	011	B	01
		1	111	D	10

Problem 10.47

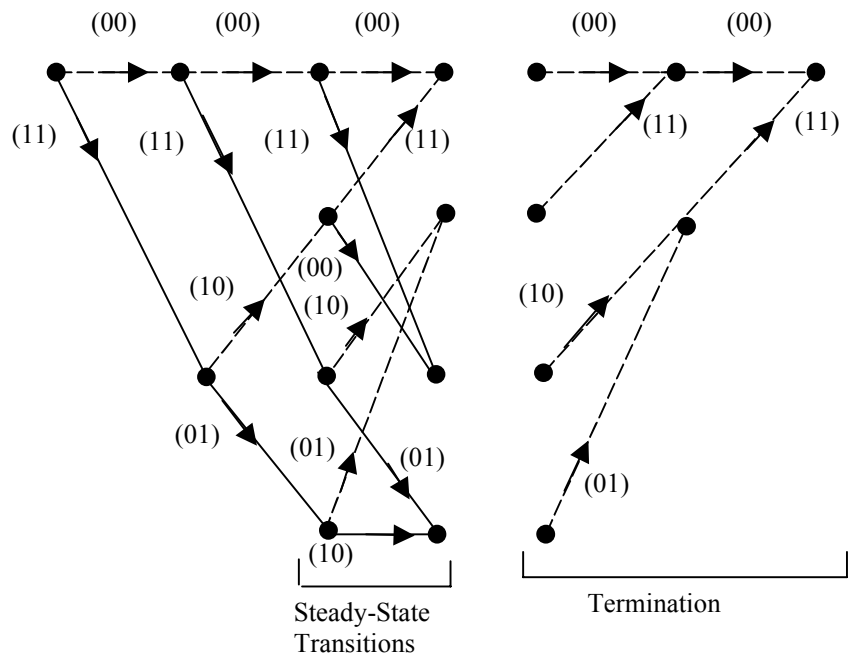

Figure 10.13:

The source rate is $r_s = 2000$ symbols/second. The channel symbol rate is

$$r_c = r_s \frac{n}{k} = 2000 \left(\frac{7}{4}\right) = 3500$$

If the burst duration is T_b, then

$$r_c T_b = r_s T_b \frac{n}{k}$$

channel symbols will be affected. Assuming a $(7, 4)$ block code, we have

$$r_c T_b = (2000)\,(0.15) \left(\frac{7}{4}\right) = 525$$

symbols affected by the burst. Let the table contain $l = 525$ codewords. This corresponds to $(525)\,(7) = 3675$ symbols in the table. Thus $3675 - 525 = 3150$ must be transmitted without error. The transmission time for 3150 symbols is

$$\frac{3150 \text{ symbols}}{3500 \text{ symbols/second}} = 0.9 \text{ seconds}$$

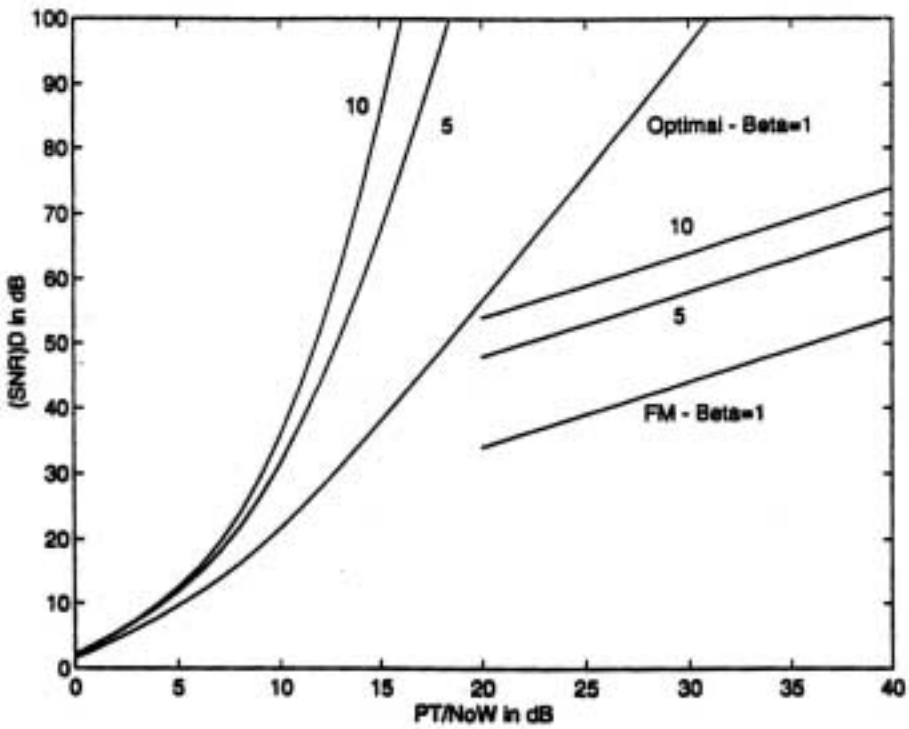

Figure 10.14:

Problem 10.48
The required plot is shown in Figure 10.14.

10.2 Computer Exercises

Computer Exercise 10.1
For two messages the problem is easy since there is only a single independent probability. The MATLAB program follows

```
a = 0.001:0.001:0.999;
```

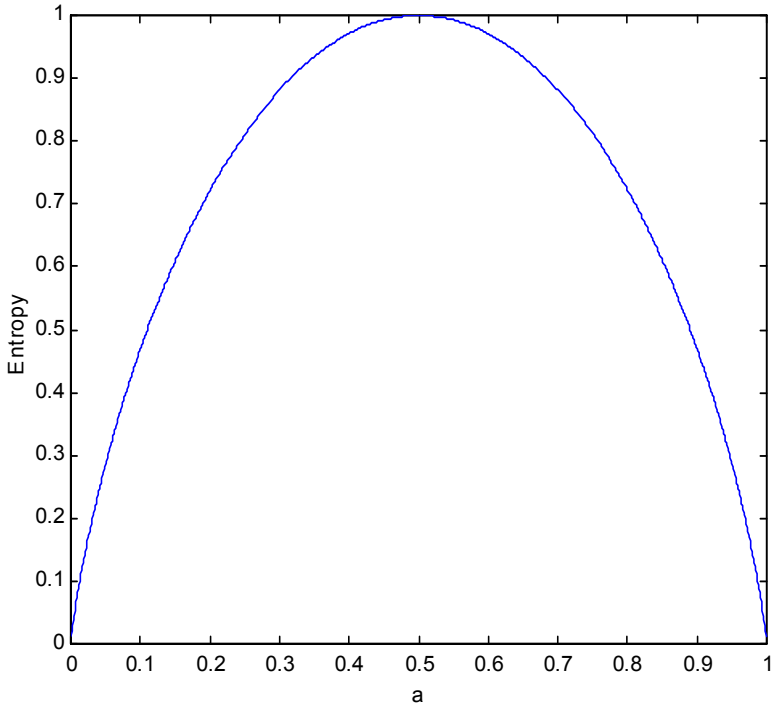

Figure 10.15:

```
b =1-a;
lna = length(a);
H = zeros(1,lna);
H=-(a.*log(a)+b.*log(b))/log(2);
plot(a,H);
xlabel('a')
ylabel('Entropy')
```

The plot is illustrated in Figure 10.15.

For three source messages we have two independent probabilities. Two of them will be specified and the remaining probability can then be determined. The resulting MATLAB code follows:

```
n = 50;                              % number of points in vector
nn = n+1;                           % increment for zero
```

```
H = zeros(nn,nn);                    % initialize entropy array
z = (0:n)/n;                         % probability vector
z(1) = eps;                          % prevent problems with log(0)
for i=1:nn
    for j=1:nn
        c = 1-z(i)-z(j);                % calculate third probability
        if c>0
            xx = [z(i) z(j) c];
            H(i,j)= -xx*log(xx)'/log(2);  % compute entropy
        end
    end
end
colormap([0 0 0])                    % b&w colormap for mesh
mesh(z,z,H);                         % plot entropy
view(-45,30);                        % set view angle
xlabel('Probability of message a')
ylabel('Probability of message b')
zlabel('Entropy')
figure                               % new figure
contour(z,z,H,10)                    % plot contour map
xlabel('Probability of message a')
ylabel('Probability of message b')
```

The resulting plot is illustrated in Figure 10.16. One should experiment with various viewing locations.

Next we draw a countour map with 10 countour lines. The result is illustrated in Figure 10.17. Note the peak at 0.33, 0.33, indicating that the maximum entropy occurs when all three source messages occur with equal probability.

Computer Exercise 10.2

The performance curves for a (15,11) Hamming code, on the basis of word energy and word error probability were derived in Problem, 10.42. The equivalent results for a (31,26) Hamming code are given in Figure 10.18. As in the case of the (7,4) Hamming code, the solid curves represent the symbol error probability with the coded case on top (worse performance) and the uncoded case on bottom. The dashed curves represent the word error probability with the coded case (better performance) on bottom.

The computer code for the (31,26) code is as follows:

```
n = 31; k = 26;
zdB = 0:10;                         % Es/No in dB
```

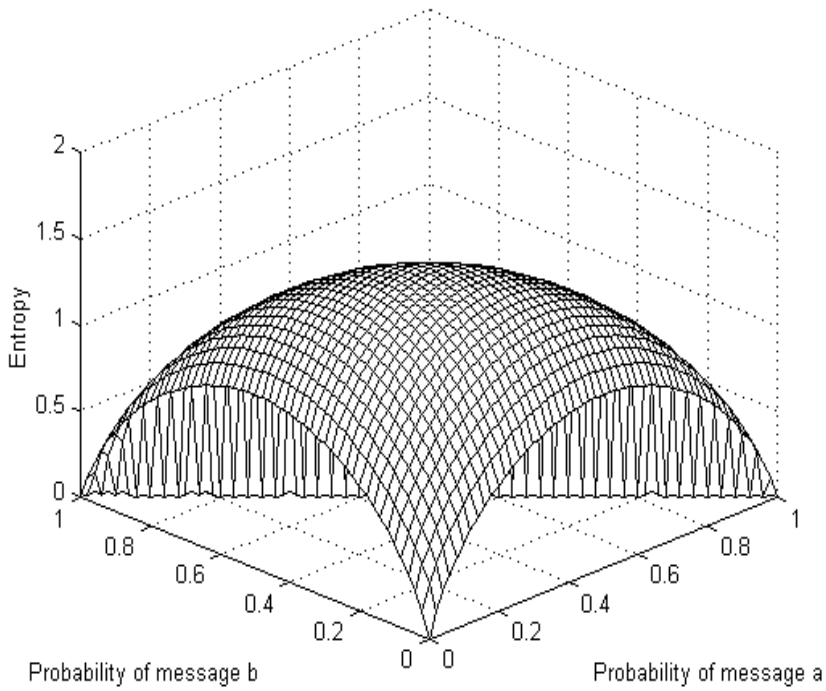

Figure 10.16:

```
z = 10.^(zdB/10);                  % Es/No in linear units
arg1 = sqrt(2*z);                  % Q-function argument for PSK
psu = 0.5*(1-erf(arg1/sqrt(2.0))); % PSK symbol error probability
pwu = 1-(1-psu).^k;                % uncoded word error probability
arg2 = sqrt(2*k*z/n);              % Q-function argument for coded PSK
psc = 0.5*(1-erf(arg2/sqrt(2.0))); % coded PSK symbol error probability
pwc = werrorc(n,k,1,psc);          % coded word error probability
zz = zdB+10*log10(k);              % scale axis for plot (STw/No)
%
semilogy(zz,psu,'k-',zz,psc,'k-',zz,pwu,'k--',zz,pwc,'k--')
xlabel('STw/No in dB')
ylabel('Probability')
```

The results for the (7,4) Hamming code are generated by changing the first line of the program to `n = 7; k = 4`. The main program calls two functions `werrorc.m` and `nkchoose.m`.

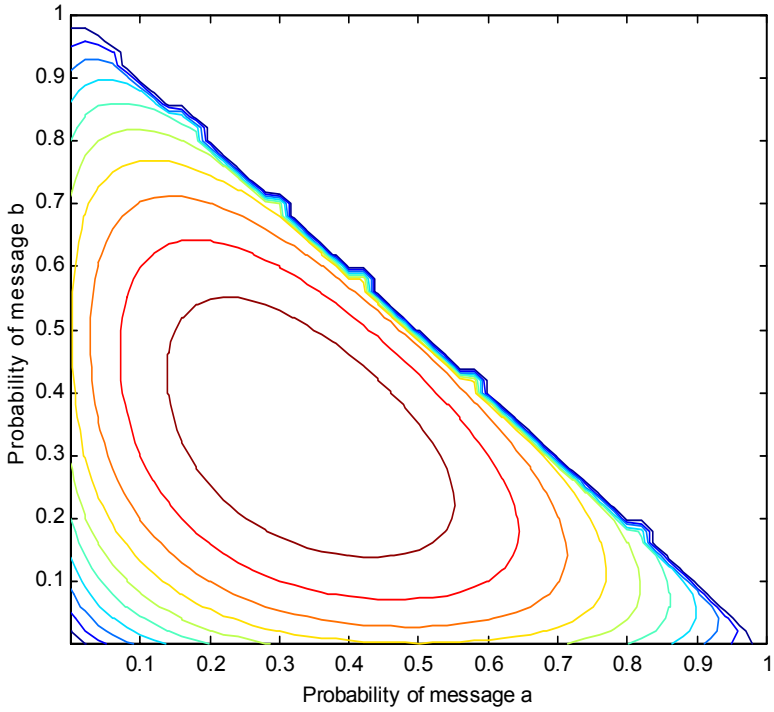

Figure 10.17:

These follow.

```
function [y] = werrorc(n,k,t,psc)
lx = length(psc);
y = zeros(1,lx);            % initialize y
for i=1:lx                  % do for each element of psc
    sum = 0;
    for j=(t+1):n
        nkc = nkchoose(n,j);
        term = log(nkc)+j*log(psc(i))+(n-j)*log(1-psc(i));
        sum = sum+exp(term);
    end
    y(i) = sum;
end
```

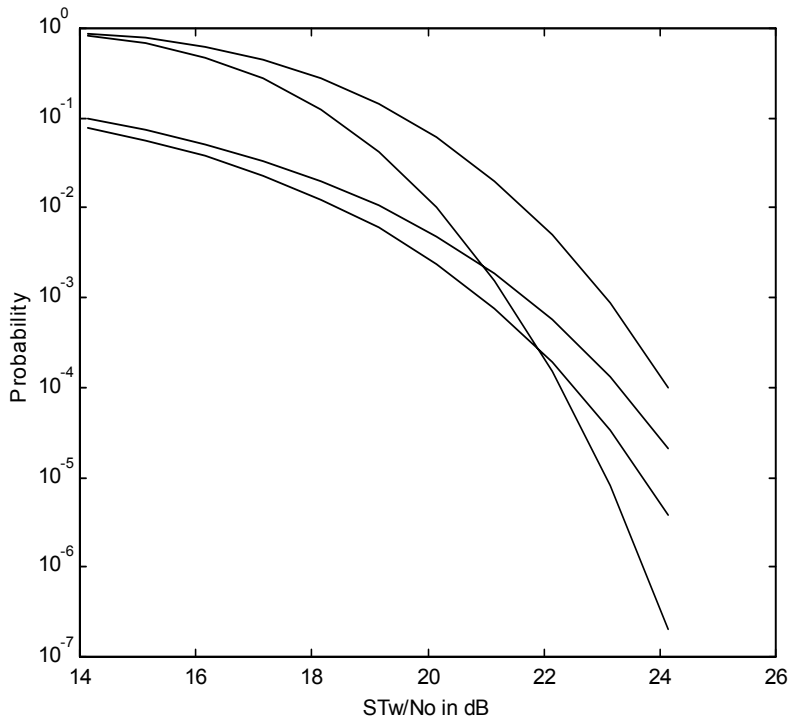

Figure 10.18:

```
function out=nkchoose(n,k)
% Computes n!/k!/(n-k)!
a = sum(log(1:n));                % ln of n!
b = sum(log(1:k));                % ln of k!
c = sum(log(1:(n-k)));            % ln of (n-k)!
out = round(exp(a-b-c));          % result
% End of function file.
```

Note: The preceding function is called `nkchoose` rather than `nchoosek` in order to avoid conflict with the function `nchoosek` in the communications toolbox. We prefer to compute the binomial coefficient using logorithms in order to obtain the large dynamic range needed for some very long codes.

Computer Exercise 10.3
The MATLAB code for implementing the given equation is

```
function [ber] = cer2ber(q,n,d,t,ps)
% Converts channel symbol error rate to decoded BER.
% Based on (1.7.30) in ''Principles of Secure Communication Systems''
% by D.J. Torrieri (2nd Edition), Artech House, 1992.
% Parameters:
% q - alphabet size (2 for binary)
% n - block length
% d - minimum distance of code
% t - correctable errors per block (usually t=(d-1)/2)
% ps - vector of channel symbol error rates
% ber - vector of bit error rates
%
lnps = length(ps);              % length of error vector
ber = zeros(1,lnps);            % initialize output vector
for k=1:lnps                    % iterate error vector
    cer = ps(k); % channel symbol error rate
    sum1 = 0; sum2 = 0; % initialize sums
% first loop evaluates first sum
    for i=(t+1):d
        term = nkchoose(n,i)*(cer^i)*((1-cer))^(n-i);
        sum1 = sum1+term;
    end
% second loop evaluates second sum
    for i=(d+1):n
        term = i*nkchoose(n,i)*(cer^i)*((1-cer)^(n-i));
        sum2 = sum2+term;
    end
% comput BER (output)
    ber(k) = (q/(2*(q-1)))*((d/n)*sum1+(1/n)*sum2);
end
```

In order to demonstrate the preceding function with a (7,4) Hamming code, we use the following MATLAB program:

```
zdB = 0:0.1:10;                   % set Eb/No axis in dB
z = 10.^(zdB/10);                 % convert to linear scale
beru = qfn(sqrt(2*z));            % PSK result
cerham = qfn(sqrt(4*z*2/7));      % CSER for (7,4) Hamming code
berham = cer2ber(2,7,3,1,cerham); % BER for Hamming code
semilogy(zdB,beru,zdB,berham)     % plot results
```

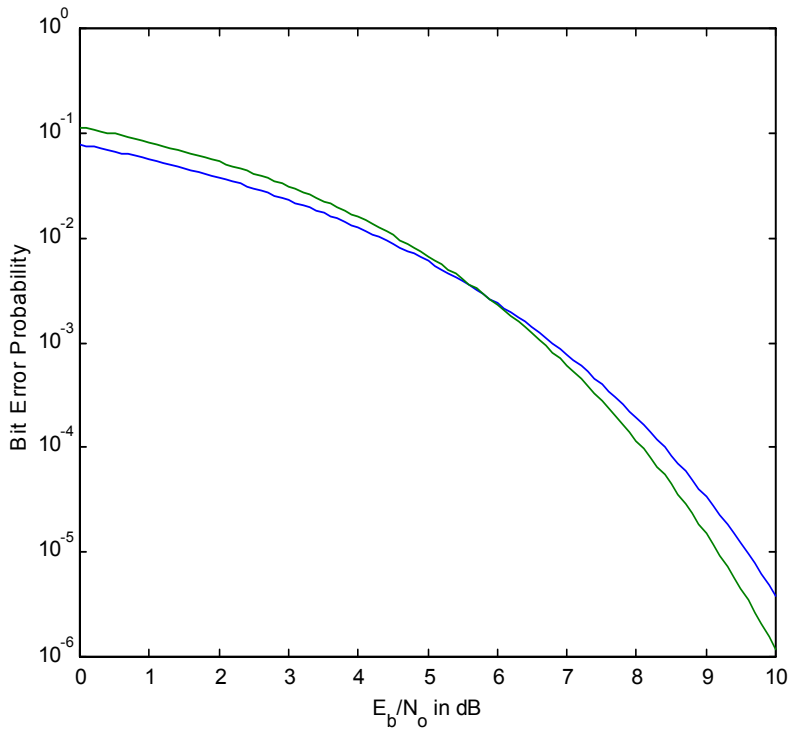

Figure 10.19:

```
xlabel('E_b/N_o in dB')           % label x axis
ylabel('Bit Error Probability')   % label y axis
```

The MATLAB function for the Gaussian Q-function follows:

```
function out=qfn(x)
% Gaussian Q function
out =0.5*erfc(x/sqrt(2));
```

Executing the code yields the result illustrated in Figure 10.19. It can be seen that the (7,4) Hamming code yields only a moderate performance improvement at high SNR.

Computer Exercise 10.4
The MATLAB program is

```
zdB = 0:0.1:10;                      % set Eb/No axis in dB
```

```
z = 10.^(zdB/10);                    % convert to linear scale
ber1 = qfn(sqrt(2*z));                 % PSK result
ber2 = qfn(sqrt(12*2*z/23));           % CSER for (23,12) Golay code
ber3 = qfn(sqrt(4*z*2/7));             % CSER for (15,11) Hamming code
bergolay = cer2ber(2,23,7,3,ber2);   % BER for Golay code
berhamming = cer2ber(2,7,3,1,ber3);  % BER for Hamming code
semilogy(zdB,ber1,zdB,bergolay,zdB,berhamming)
xlabel('E_b/N_o in dB')
ylabel('Bit Error Probability')
```

The funcrions Q and cer2ber are given in the previous computer exercise.

The results are illustrated in Figure 10.20. The order of the curves, for high signal-to-noise ratio, are in order of the error correcting capability. The top curve (worst performance) is the uncoded case. The middle curve is for the (7,4) Hamming code and the bottom curve is for the (23,12) Golay code.

Computer Exercise 10.5

The MATLAB code for generating the performance curves illustrated in Figure 10.18 in the text follows.

```
zdB = 0:30;
z = 10.^(zdB/10);
qa = 0.5*exp(-0.5*z);
qa3 = 0.5*exp(-0.5*z/3);
qa7 = 0.5*exp(-0.5*z/7);
pa7 = 35*((1-qa7).^3).*(qa7.^4)+21*((1-qa7).^2).*(qa7.^5)...
      +7*(1-qa7).*(qa7.^6)+(qa7.^7);
pa3 = 3*(1-qa3).*(qa3.^2)+(qa3.^3);
qr = 0.5./(1+z/2);
qr3 = 0.5./(1+z/(2*3));
qr7 = 0.5./(1+z/(2*7));
pr7 = 35*((1-qr7).^3).*(qr7.^4)+21*((1-qr7).^2).*(qr7.^5)...
      +7*(1-qr7).*(qr7.^6)+(qr7.^7);
pr3 = 3*(1-qr3).*(qr3.^2)+(qr3.^3);
semilogy(zdB,qr,zdB,pr3,zdB,pr7,zdB,qa,zdB,pa3,zdB,pa7)
axis([0 30 0.0001 1])
xlabel('Signal-to-Noise Ratio, z - dB')
ylabel('Probability ')
```

Executing the code yields the results illustrated in Figure 10.21. The curves can be identified from Figure 10.18 in the text.

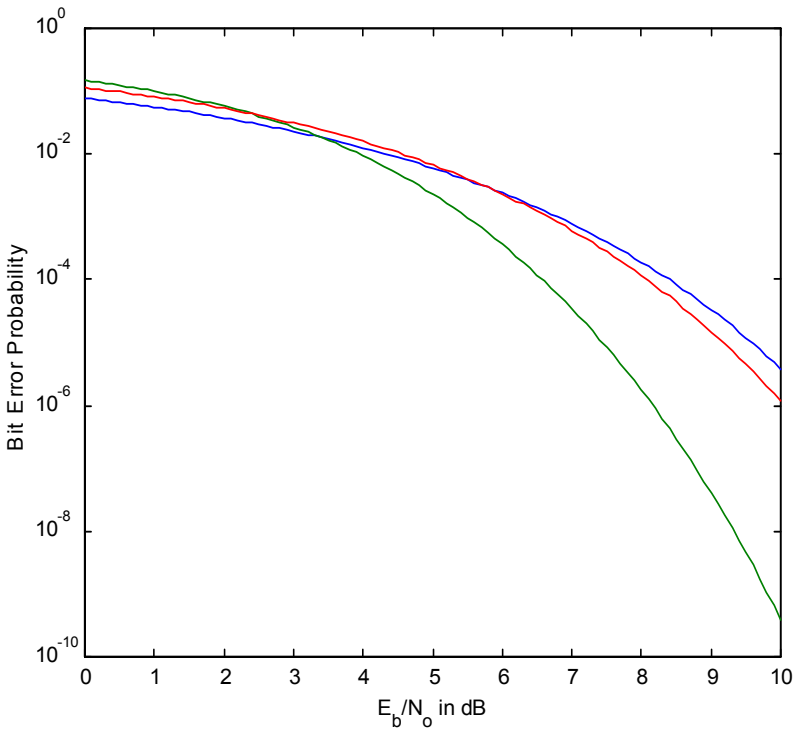

Figure 10.20:

Computer Exercise 10.6

For this Computer Exercise we use the uncoded Rayleigh fading result from the previous Computer Exercise and also use the result for a Rayleigh fading channel with a rate 1/7 repetition code. The new piece of information needed for this Computer Exercise is the result from Problem 9.17. The MATLAB code follows:

```
zdB = 0:30;
z = 10.^(zdB/10);
qr1 = 0.5./(1+z/2);
qr7 = 0.5./(1+z/(2*7));
pr7 = 35*((1-qr7).^3).*(qr7.^4)+21*((1-qr7).^2).*(qr7.^5)...
      +7*(1-qr7).*(qr7.^6)+(qr7.^7);
% Calculations for diversity system
pd7 = zeros(1,31);
N = 7;
```

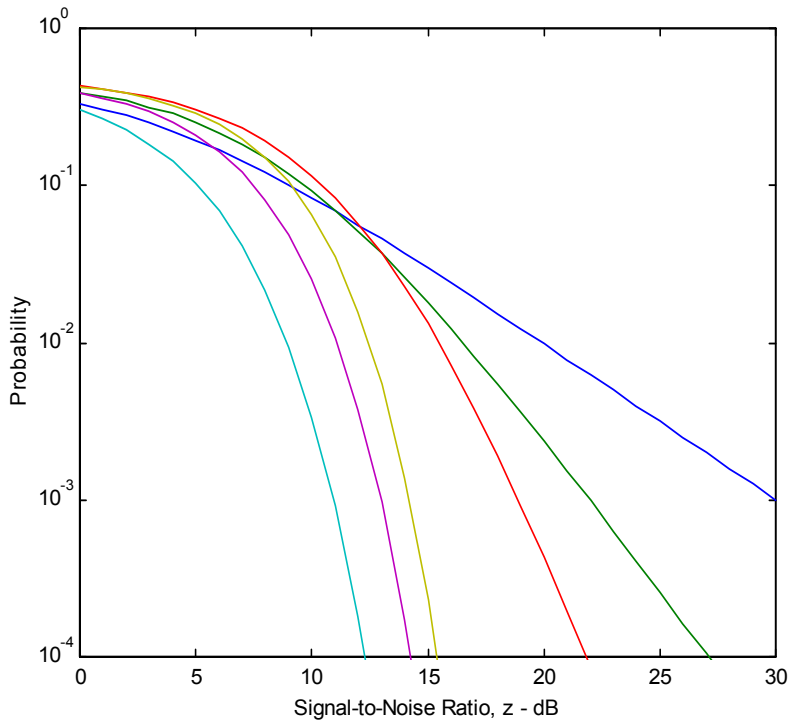

Figure 10.21:

```
for i=1:31
    sum = 0;
    a = 0.5/(1+z(i)/(2*N));
    for j=0:(N-1)
        term = nkchoose(N+j-1,j)*((1-a)^j);
        sum = sum+term;
    end
    pd7(i) = (a^N)*sum;
end
%
semilogy(zdB,qr1,zdB,pr7,zdB,pd7)
axis([0 30 0.0001 1])
xlabel('Signal-to-Noise Ratio, z - dB')
ylabel('Probability')
```

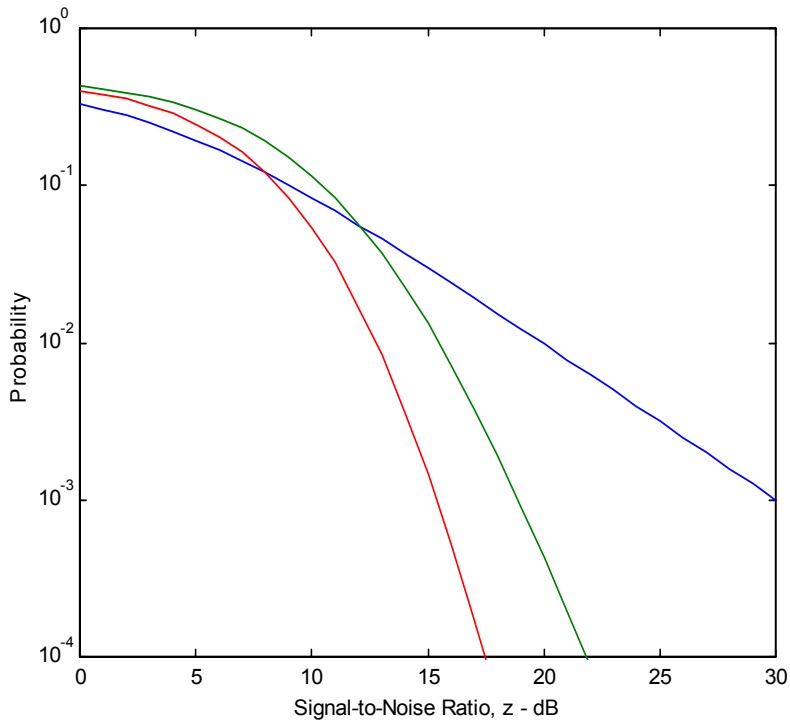

Figure 10.22:

The function nkchoose follows:

```
function out=nkchoose(n,k)
% Computes n!/k!/(n-k)!
a = sum(log(1:n));          % ln of n!
b = sum(log(1:k));          % ln of k!
c = sum(log(1:(n-k)));      % ln of (n-k)!
out = round(exp(a-b-c));    % result
% End of function file.
```

Executing the MATLAB program yields the result illustrated in Figure 10.22. The curves can be identified by comparison with Figure 10.20 inthe text.

Appendix A

Physical Noise Sources and Noise Calculations

Problem A.1

All parts of the problem are solved using the relation

$$V_{\text{rms}} = \sqrt{4kTRB}$$

where

$$\begin{aligned} k &= 1.38 \times 10^{-23} \text{ J/K} \\ B &= 30 \text{ MHz} = 3 \times 10^{7} \text{ Hz} \end{aligned}$$

(a) For $R = 10,000$ ohms and $T = T_0 = 290$ K

$$\begin{aligned} V_{\text{rms}} &= \sqrt{4\,(1.38 \times 10^{-23})\,(290)\,(10^4)\,(3 \times 10^7)} \\ &= 6.93 \times 10^{-5} \text{ V rms} \\ &= 69.3\ \mu\text{V rms} \end{aligned}$$

(b) V_{rms} is smaller than the result in part (a) by a factor of $\sqrt{10} = 3.16$. Thus

$$V_{\text{rms}} = 21.9\ \mu\text{V rms}$$

(c) V_{rms} is smaller than the result in part (a) by a factor of $\sqrt{100} = 10$. Thus

$$V_{\text{rms}} = 6.93\ \mu\text{V rms}$$

(d) Each answer becomes smaller by factors of 2, $\sqrt{10} = 3.16$, and 10, respectively.

Problem A.2

Use

$$I = I_s \left[\exp\left(\frac{eV}{kT}\right) - 1 \right],$$

We want $I > 20I_s$ or $\exp\left(\frac{eV}{kT}\right) - 1 > 20$.
(a) At $T = 290$ K, $\frac{e}{kT} \simeq 40$, so we have $\exp(40V) > 21$ giving

$$\begin{aligned}
V &> \frac{\ln(21)}{40} = 0.0761 \text{ volts} \\
i_{\text{rms}}^2 &\simeq 2eIB \simeq 2eBI_s \exp\left(\frac{eV}{kT}\right) \\
\text{or } \frac{i_{\text{rms}}^2}{B} &= 2eI_s \exp\left(\frac{eV}{kT}\right) \\
&= 2\left(1.6 \times 10^{-19}\right)\left(1.5 \times 10^{-5}\right) \exp(40 \times 0.0761) \\
&= 1.0075 \times 10^{-22} \text{ A}^2/\text{Hz}
\end{aligned}$$

(b) If $T = 29$ K, then $\frac{e}{kT} = 400$, and for $I > 20I_s$, we need $\exp(400V) > 21$ or

$$V > \frac{\ln(21)}{400} = 7.61 \times 10^{-3} \text{ volts}$$

Thus

$$\begin{aligned}
\frac{i_{\text{rms}}^2}{B} &= 2\left(1.6 \times 10^{-19}\right)\left(1.5 \times 10^{-5}\right) \exp\left(400 \times 7.61 \times 10^{-3}\right) \\
&= 1.0075 \times 10^{-22} \text{ A}^2/\text{Hz}
\end{aligned}$$

as before.

Problem A.3

(a) Use Nyquist's formula to get the equivalent circuit of R_{eq} in parallel with R_3, where R_{eq} is given by

$$R_{\text{eq}} = \frac{R_L (R_1 + R_2)}{R_1 + R_2 + R_L}$$

The noise equivalent circuit is then as shown in Figure A.1. The equivalent noise voltages are

$$\begin{aligned}
V_1 &= \sqrt{4kTR_{\text{eq}}B} \\
V_2 &= \sqrt{4kTR_3B}
\end{aligned}$$

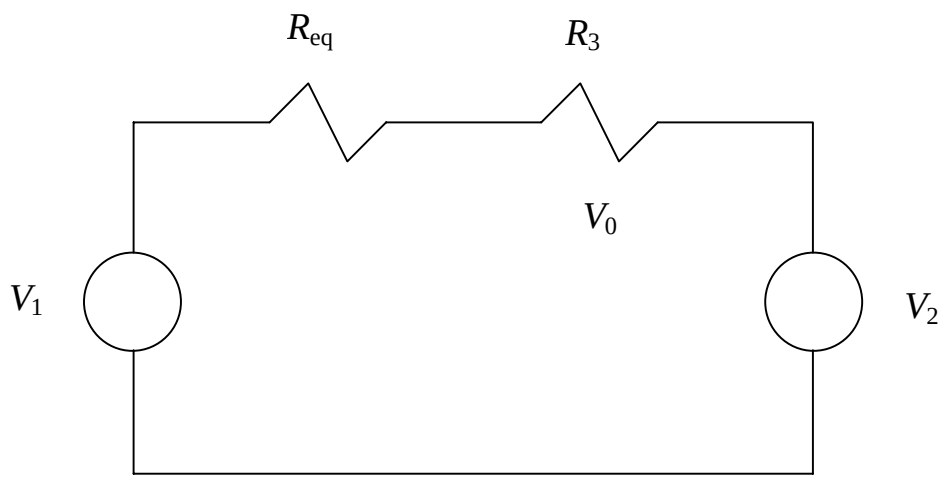

Figure A.1:

Adding noise powers to get V_0 we obtain

$$\begin{aligned} V_0^2 &= \left(\frac{R_3}{R_{\text{eq}} + R_3}\right)^2 \left(V_1^2 + V_2^2\right) \\ &= \left(\frac{R_3}{R_{\text{eq}} + R_3}\right)^2 (4kTB)\,(R_{\text{eq}} + R_3) \\ &= \frac{(4kTB)\,R_3^2}{(R_{\text{eq}} + R_3)} \end{aligned}$$

(b) With $R_1 = 2000\ \Omega,\ R_2 = R_L = 300\ \Omega$, and $R_3 = 500\ \Omega$, we have

$$R_{\text{eq}} = \frac{300\,(2000 + 300)}{300 + 2000 + 300} = 265.4\ \Omega$$

and

$$\begin{aligned} \frac{V_0^2}{B} &= \frac{(4kT)\,R_3^2}{(R_{\text{eq}} + R_3)} \\ &= \frac{4\left(1.38 \times 10^{-23}\right)(290)\,(500)^2}{500 + 265.4} \\ &= 5.23 \times 10^{-18}\ \text{V}^2/\text{Hz} \end{aligned}$$

Problem A.4

Find the equivalent resistance for the R_1, R_2, R_3 combination and set R_L equal to this to get

$$R_L = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3}$$

Problem A.5

Using Nyquist's formula, we find the equivalent resistance looking backing into the terminals with V_{rms} across them. It is

$$\begin{aligned} R_{\text{eq}} &= 50\text{k} \parallel 20\text{ k} \parallel (5\text{ k} + 10\text{ k} + 5\text{ k}) \\ &= 50\text{k} \parallel 20\text{ k} \parallel 20\text{ k} \\ &= 50\text{k} \parallel 10\text{ k} \\ &= \frac{(50\text{k})\,(10\text{ k})}{50\text{k} + 10\text{ k}} \\ &= 8,333\ \Omega \end{aligned}$$

Thus

$$\begin{aligned} V_{\text{rms}}^2 &= 4kTR_{\text{eq}}B \\ &= 4\left(1.38 \times 10^{-23}\right)(400)\,(8333)\left(2 \times 10^{6}\right) \\ &= 3.68 \times 10^{-10}\ \text{V}^2 \end{aligned}$$

which gives

$$V_{\text{rms}} = 19.18\ \mu\text{V rms}$$

Problem A.6

To find the noise figure, we first determine the noise power due to a source at the output, and then due to the source and the network. Initally assume unmatched conditions. The results are

$$\overline{V_0^2}_{\text{ due to } R_S\text{, only}} = \left(\frac{R_2 \parallel R_L}{R_S + R_1 + R_2 \parallel R_L}\right)^2 (4kTR_SB)$$

$$\begin{aligned} \overline{V_0^2}_{\text{ due to } R_1 \text{ and } R_2} = &\left(\frac{R_2 \parallel R_L}{R_S + R_1 + R_2 \parallel R_L}\right)^2 (4kTR_1B) \\ &+ \left(\frac{R_L \parallel (R_1 + R_S)}{R_2 + (R_1 + R_S) \parallel R_L}\right)^2 (4kTR_2B) \end{aligned}$$

$$\overline{V_0^2}_{\text{due to } R_S,\, R_1 \text{ and } R_2} = \left(\frac{R_2 \parallel R_L}{R_S + R_1 + R_2 \parallel R_L}\right)^2 \left[4kT\left(R_S + R_1\right)B\right] + \left(\frac{R_L \parallel (R_1 + R_S)}{R_2 + (R_1 + R_S) \parallel R_L}\right)^2 \left(4kTR_2B\right)$$

The noise figure is

$$F = 1 + \frac{R_1}{R_S} + \left(\frac{R_L \parallel (R_1 + R_S)}{R_2 + (R_1 + R_S) \parallel R_L}\right)^2 \left(\frac{R_S + R_1 + R_2 \parallel R_L}{R_2 \parallel R_L}\right)^2 \frac{R_2}{R_S}$$

In the above,

$$R_a \parallel R_b = \frac{R_a R_b}{R_a + R_b}$$

Note that the noise due to R_L has been excluded because it belongs to the next stage. Since this is a matching circuit, we want the input matched to the source and the output matched to the load. Matching at the input requires that

$$R_S = R_{\text{in}} = R_1 + R_2 \parallel R_L = R_1 + \frac{R_2 R_L}{R_2 + R_L}$$

and matching at the output requires that

$$R_L = R_{\text{out}} = R_2 \parallel (R_1 + R_S) = \frac{R_2\left(R_1 + R_S\right)}{R_1 + R_2 + R_S}$$

Next, these expressions are substituted back into the expression for F. After some simplification, this gives

$$F = 1 + \frac{R_1}{R_S} + \left(\frac{2R_L^2 R_S\left(R_1 + R_2 + R_S\right)/\left(R_S - R_1\right)}{R_2^2\left(R_1 + R_S + R_L\right) + R_L^2\left(R_1 + R_2 + R_S\right)}\right)^2 \frac{R_2}{R_S}$$

Note that if $R_1 >> R_2$ we then have matched conditions of $R_L = R_2$ and $R_S = R_1$. Then, the noise figure simplifies to

$$F = 2 + 16\frac{R_1}{R_2}$$

Note that the simple resistive pad matching circuit is very poor from the standpoint of noise. The equivalent noise temperature is found by using

$$\begin{aligned} T_e &= T_0\left(F - 1\right) \\ &= T_0\left(1 + 16\frac{R_1}{R_2}\right) \end{aligned}$$

Problem A.7

The important relationships are

$$F_{\mathsf{I}} = 1 + \frac{T_{\mathsf{e_I}}}{T_{\mathsf{0}}}$$

$$T_{\mathsf{e_I}} = T_{\mathsf{0}}\,(F_{\mathsf{I}} - 1)$$

$$T_{\mathsf{e_0}} = T_{\mathsf{e_1}} + \frac{T_{\mathsf{e_2}}}{G_{\mathsf{a_1}}} + \frac{T_{\mathsf{e_3}}}{G_{\mathsf{a_1}} G_{\mathsf{a_2}}}$$

Completion of the table gives

Ampl. No.	F	$T_{\mathsf{e_i}}$	$G_{\mathsf{a_i}}$
1	not needed	300 K	10 dB = 10
2	6 dB	864.5 K	30 dB = 1000
3	11 dB	3360.9 K	30 dB = 1000

Therefore,

$$\begin{aligned} T_{\mathsf{e_0}} &= 300 + \frac{864.5}{10} + \frac{3360.9}{(10)\,(1000)} \\ &= 386.8 \text{ K} \end{aligned}$$

Hence,

$$\begin{aligned} F_{\mathsf{0}} &= 1 + \frac{T_{\mathsf{e_0}}}{T_{\mathsf{0}}} \\ &= 2.33 = 3.68 \text{ dB} \end{aligned}$$

(b) With amplifiers 1 and 2 interchanged

$$\begin{aligned} T_{\mathsf{e_0}} &= 864.5 + \frac{300}{10} + \frac{3360.9}{(10)\,(1000)} \\ &= 865.14 \text{ K} \end{aligned}$$

This gives a noise figure of

$$\begin{aligned} F_{\mathsf{0}} &= 1 + \frac{865.14}{290} \\ &= 3.98 = 6 \text{ dB} \end{aligned}$$

(c) See part (a) for the noise temperatures.

(d) For $B = 50$ kHz, $T_S = 1000$ K, and an overall gain of $G_a = 10^7$, we have, for the configuration of part (a)

$$\begin{aligned} P_{na,\,\text{out}} &= G_a k\,(T_0 + T_{e_0})\,B \\ &= 10^7 \left(1.38 \times 10^{-23}\right)(1000 + 386.8)\left(5 \times 10^4\right) \\ &= 9.57 \times 10^{-9} \text{ watt} \end{aligned}$$

We desire

$$\frac{P_{sa,\,\text{out}}}{P_{na,\,\text{out}}} = 10^4 = \frac{P_{sa,\,\text{out}}}{9.57 \times 10^{-9}}$$

which gives

$$P_{sa,\,\text{out}} = 9.57 \times 10^{-5} \text{ watt}$$

For part (b), we have

$$\begin{aligned} P_{na,\,\text{out}} &= 10^7 \left(1.38 \times 10^{-23}\right)(1000 + 865.14)\left(5 \times 10^4\right) \\ &= 1.29 \times 10^{-8} \text{ watt} \end{aligned}$$

Once again, we desire

$$\frac{P_{sa,\,\text{out}}}{P_{na,\,\text{out}}} = 10^4 = \frac{P_{sa,\,\text{out}}}{1.29 \times 10^{-8}}$$

which gives

$$P_{sa,\,\text{out}} = 1.29 \times 10^{-4} \text{ watt}$$

and

$$P_{sa,\,\text{in}} = \frac{P_{sa,\,\text{out}}}{G_a} = 1.29 \times 10^{-11} \text{ watt}$$

Problem A.8

(a) The noise figure of the cascade is

$$F_{\text{overall}} = F_1 + \frac{F_2 - 1}{G_{a_1}} = L + \frac{F - 1}{(1/L)} = LF$$

(b) For two identicalattenuator-amplifier stages

$$F_{\text{overall}} = L + \frac{F-1}{(1/L)} + \frac{L-1}{(1/L)\,L} + \frac{F-1}{(1/L)\,L\,(1/L)} = 2LF - 1 \approx 2LF, \quad L >> 1$$

(c) Generalizing, for N stages we have

$$F_{\text{overall}} \approx NFL$$

Problem A.9

The data for this problem is

Stage	F_{i}	G_{i}
1 (preamp)	2 dB = 1.58	G_1
2 (mixer)	8 dB = 6.31	1.5 dB = 1.41
3 (amplifier)	5 dB = 3.16	30 dB = 1000

The overall noise figure is

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$$

which gives

$$5 \text{ dB} = 3.16 \geq 1.58 + \frac{6.31 - 1}{G_1} + \frac{3.16 - 1}{1.42 G_1}$$

or

$$G_1 \geq \frac{5.31 - (2.16/1.41)}{1.58} = 4.33 = 6.37 \text{ dB}$$

(b) First, assume that $G_1 = 15$ dB $= 31.62$. Then

$$\begin{aligned} F &= 1.58 + \frac{6.31 - 1}{31.62} + \frac{3.16 - 1}{(1.41)(31.62)} \\ &= 1.8 = 2.54 \text{ dB} \end{aligned}$$

Then

$$\begin{aligned} T_{\mathsf{es}} &= T_0 (F - 1) \\ &= 290 (1.8 - 1) \\ &= 230.95 \text{ K} \end{aligned}$$

and

$$\begin{aligned} T_{\text{overall}} &= T_{\mathsf{es}} + T_{\mathsf{a}} \\ &= 230.95 + 300 \\ &= 530.95 \text{ K} \end{aligned}$$

Now use G_1 as found in part (a):

$$\begin{aligned} F &= 3.16 \\ T_{\mathsf{es}} &= 290 (3.16 - 1) = 626.4 \text{ K} \\ T_{\text{overall}} &= 300 + 626.4 = 926.4 \text{ K} \end{aligned}$$

(c) For $G_1 = 15$ dB $= 31.62$, $G_a = 31.62\,(1.41)\,(1000) = 4.46 \times 10^4$. Thus

$$\begin{aligned} P_{na,\,\mathrm{out}} &= {}^{\mathsf{i}}4.46 \times 10^4{}^{¢}\,{}^{\mathsf{i}}1.38 \times 10^{-23}{}^{¢}\,(530.9)\,{}^{\mathsf{i}}10^9{}^{¢} \\ &= 3.27 \times 10^{-9} \text{ watts} \end{aligned}$$

For $G_1 = 6.37$ dB $= 4.33$, $G_a = 4.33\,(1.41)\,(1000) = 6.11 \times 10^3$. Thus

$$\begin{aligned} P_{na,\,\mathrm{out}} &= {}^{\mathsf{i}}6.11 \times 10^3{}^{¢}\,{}^{\mathsf{i}}1.38 \times 10^{-23}{}^{¢}\,(926.4)\,{}^{\mathsf{i}}10^7{}^{¢} \\ &= 7.81 \times 10^{-10} \text{ watts} \end{aligned}$$

Note that for the second case, we get less noise power out even wth a larger T_{overall}. This is due to the lower gain of stage 1, which more than compensates for the larger input noise power.

(d) A transmission line with loss $L = 2$ dB connects the antenna to the preamp. We first find T_S for the transmission line/preamp/mixer/amp chain:

$$F_S = F_{\mathrm{TL}} + \frac{F_1 - 1}{G_{\mathrm{TL}}} + \frac{F_2 - 1}{G_{\mathrm{TL}} G_1} + \frac{F_3 - 1}{G_{\mathrm{TL}} G_1 G_2},$$

where

$$G_{\mathrm{TL}} = 1/L = 10^{-2/10} = 0.631 \text{ and } F_{\mathrm{TL}} = L = 10^{2/10} = 1.58$$

Assume two cases for G_1: 15 dB and 6.37 dB. First, for $G_1 = 15$ dB $= 31.6$, we have

$$\begin{aligned} F_S &= 1.58 + \frac{1.58 - 1}{0.631} + \frac{6.31 - 1}{(0.631)\,(31.6)} + \frac{3.16 - 1}{(0.631)\,(31.6)\,(1.41)} \\ &= 2.84 \end{aligned}$$

This gives

$$T_S = 290\,(2.84 - 1) = 534 \text{ K}$$

and

$$T_{\mathrm{overall}} = 534 + 300 = 834 \text{ K}$$

Now, for $G_1 = 6.37$ dB $= 4.33$, we have

$$\begin{aligned} F_S &= 1.58 + \frac{1.58 - 1}{0.631} + \frac{6.31 - 1}{(0.631)\,(4.33)} + \frac{3.16 - 1}{(0.631)\,(4.33)\,(1.41)} \\ &= 5 \end{aligned}$$

This gives

$$T_S = 290\,(5 - 1) = 1160 \text{ K}$$

and

$$T_{\text{overall}} = 1160 + 300 = 1460 \text{ K}$$

Problem A.10

(a) Using

$$P_{\mathsf{na},\,\text{out}} = G_{\mathsf{a}} k T_{\mathsf{S}} B$$

with the given values yields

$$P_{\mathsf{na},\,\text{out}} = 7.45 \times 10^{-6} \text{ watts}$$

(b) We want

$$\frac{P_{\mathsf{sa},\,\text{out}}}{P_{\mathsf{na},\,\text{out}}} = 10^5$$

or

$$P_{\mathsf{sa},\,\text{out}} = \left(10^5\right)\left(7.45 \times 10^{-6}\right) = 0.745 \text{ watts}$$

This gives

$$\begin{aligned} P_{\mathsf{sa},\,\text{in}} &= \frac{P_{\mathsf{sa},\,\text{out}}}{G_{\mathsf{a}}} = 0.745 \times 10^{-8} \text{ watts} \\ &= -51.28 \text{ dBm} \end{aligned}$$

Problem A.11

(a) For $\Delta A = 1$ dB, $Y = 1.259$ and the effective noise temperature is

$$T_{\mathsf{e}} = \frac{600 - (1.259)\,(300)}{1.259 - 1} = 858.3 \text{ K}$$

For $\Delta A = 1.5$ dB, $Y = 1.413$ and the effective noise temperature is

$$T_{\mathsf{e}} = \frac{600 - (1.413)\,(300)}{1.413 - 1} = 426.4 \text{ K}$$

For $\Delta A = 2$ dB, $Y = 1.585$ and the effective noise temperature is

$$T_{\mathsf{e}} = \frac{600 - (1.585)\,(300)}{1.585 - 1} = 212.8 \text{ K}$$

(b) These values can be converted to noise figure using

$$F = 1 + \frac{T_{\mathsf{e}}}{T_{\mathsf{0}}}$$

With $T_0 = 290$ K, we get the following values: (1) For $\Delta A = 1$ dB, $F = 5.98$ dB; (2) For $\Delta A = 1.5$ dB, $F = 3.938$ dB; (3) For $\Delta A = 2$ dB, $F = 2.39$ dB.

Problem A.12

(a) Using the data given, we can determine the following:

$$\begin{aligned} \left(\frac{\lambda}{4\pi d}\right)^2 &= 0.039 \text{ m} \\ &= -202.4 \text{ dB} \\ G_T &= 39.2 \text{ dB} \\ P_T G_T &= 74.2 \text{ dBW} \end{aligned}$$

This gives

$$P_S = -202.4 + 74.2 + 6 - 5 = -127.2 \text{ dBW}$$

(b) Using $P_n = kT_eB$ for the noise power, we get

$$\begin{aligned} P_n &= 10\log_{10}\left(kT_0\left(\frac{T_e}{T_0}\right)B\right) \\ &= 10\log_{10}[kT_0] + 10\log_{10}\left(\frac{T_e}{T_0}\right) + 10\log_{10}(B) \\ &= -174 + 10\log_{10}\left(\frac{1000}{290}\right) + 10\log_{10}\left(10^6\right) \\ &= -108.6 \text{ dBm} \\ &= -138.6 \text{ dBW} \end{aligned}$$

(c)

$$\begin{aligned} \left(\frac{P_S}{P_n}\right)_{\text{dB}} &= -127.2 - (-138.6) \\ &= 11.4 \text{ dB} \\ &= 10^{1.14} = 13.8 \text{ ratio} \end{aligned}$$

(d) Assuming the SNR $= z = E_b/N_0 = 13.8$, we get the results for various digital signaling techniques given in the table below:

Modulation type	Error probability
BPSK	$Q\left(\sqrt{2z}\right) = 7.4 \times 10^{-8}$
DPSK	$\frac{1}{2}e^{-z} = 5.06 \times 10^{-7}$
Noncoh. FSK	$\frac{1}{2}e^{-z/2} = 5.03 \times 10^{-4}$
QPSK	Same as BPSK