Solution to Chapter 3 Problems

Problem 3.1

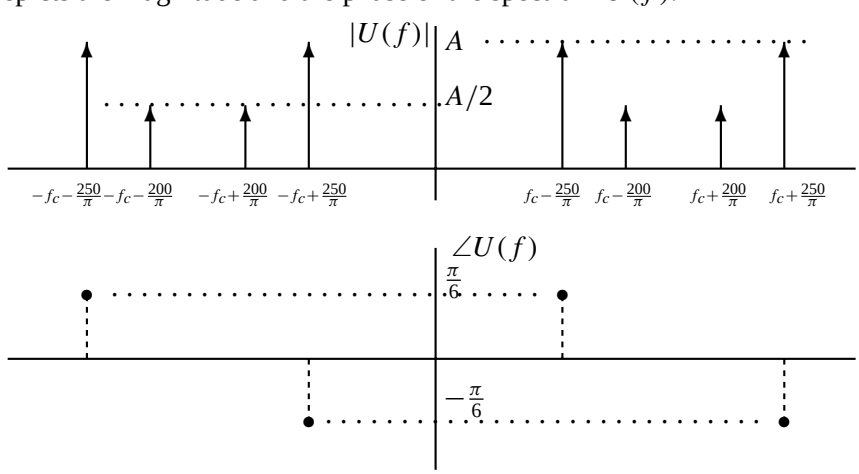
The modulated signal is

$$
\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t) \\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t + \frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t) \\
&= A\cos(2\pi(4\times 10^3 + \frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3 - \frac{200}{\pi})t) \\
&\quad +2A\sin(2\pi(4\times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3})
\end{aligned}
$$

Taking the Fourier transform of the previous relation, we obtain

$$
\begin{aligned}
U(f) &= A\left[\delta(f - \frac{200}{\pi}) + \delta(f + \frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f - \frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f + \frac{250}{\pi})\right] \\
&\quad \star\frac{1}{2}[\delta(f - 4\times 10^3) + \delta(f + 4\times 10^3)] \\
&= \frac{A}{2}\left[\delta(f - 4\times 10^3 - \frac{200}{\pi}) + \delta(f - 4\times 10^3 + \frac{200}{\pi})\right. \\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f - 4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f - 4\times 10^3 + \frac{250}{\pi}) \\
&\quad +\delta(f + 4\times 10^3 - \frac{200}{\pi}) + \delta(f + 4\times 10^3 + \frac{200}{\pi}) \\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f + 4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f + 4\times 10^3 + \frac{250}{\pi})\right]
\end{aligned}
$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



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To find the power content of the modulated signal we write $u^2(t)$ as

$$\begin{aligned} u^2(t) &= A^2\cos^2(2\pi(4\times 10^3+\frac{200}{\pi})t)+A^2\cos^2(2\pi(4\times 10^3-\frac{200}{\pi})t) \\ &\quad +4A^2\sin^2(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3})+4A^2\sin^2(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3}) \\ &\quad +\text{terms of cosine and sine functions in the first power} \end{aligned}$$

Hence,

$$P=\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)dt=\frac{A^2}{2}+\frac{A^2}{2}+\frac{4A^2}{2}+\frac{4A^2}{2}=5A^2$$

Problem 3.2

$$u(t)=m(t)c(t)=A(\text{sinc}(t)+\text{sinc}^2(t))\cos(2\pi f_c t)$$

Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U(f) &= \frac{A}{2}[\Pi(f)+\Lambda(f)]\star(\delta(f-f_c)+\delta(f+f_c)) \\ &= \frac{A}{2}[\Pi(f-f_c)+\Lambda(f-f_c)+\Pi(f+f_c)+\Lambda(f+f_c)] \end{aligned}$$

$\Pi(f-f_c)\neq 0$ for $|f-f_c|<\frac{1}{2}$, whereas $\Lambda(f-f_c)\neq 0$ for $|f-f_c|<1$. Hence, the bandwidth of the bandpass filter is 2.

Problem 3.3

The following figure shows the modulated signals for $A=1$ and $f_0=10$. As it is observed both signals have the same envelope but there is a phase reversal at $t=1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ (right plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0=3$.

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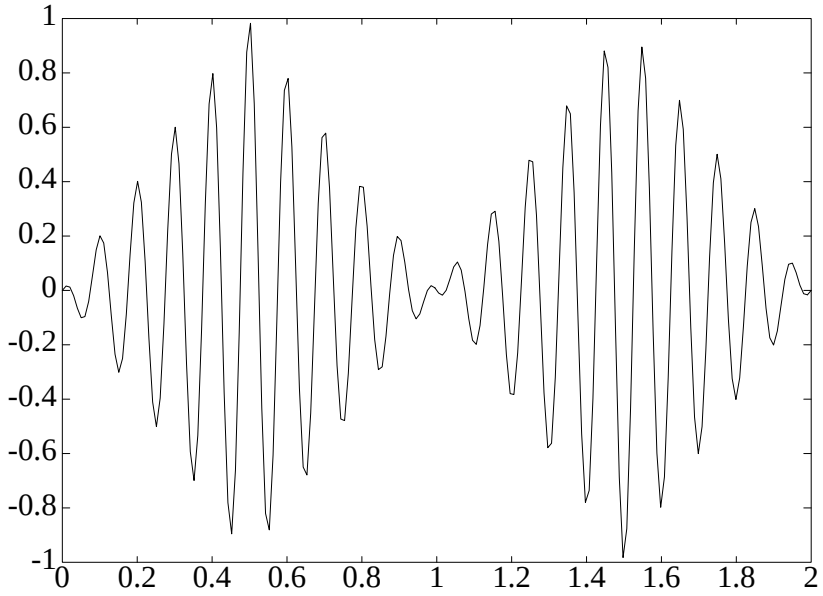

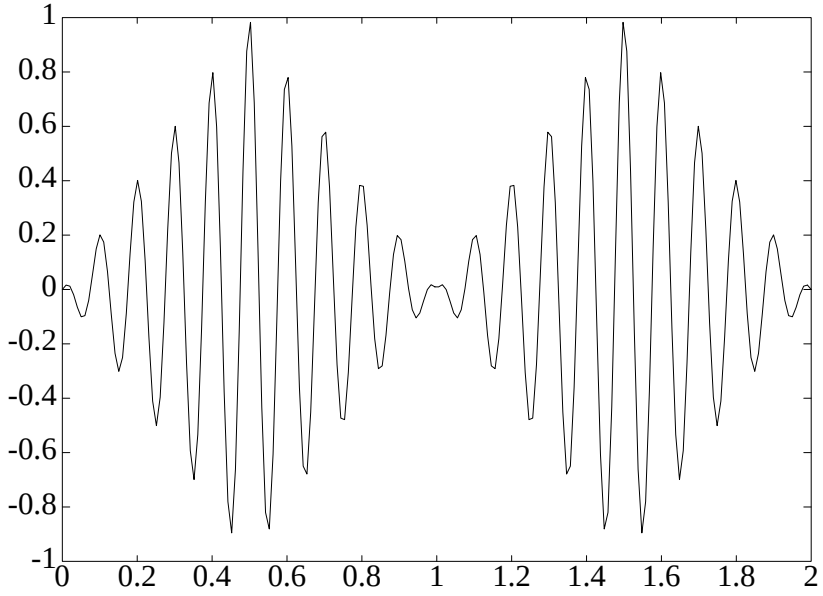

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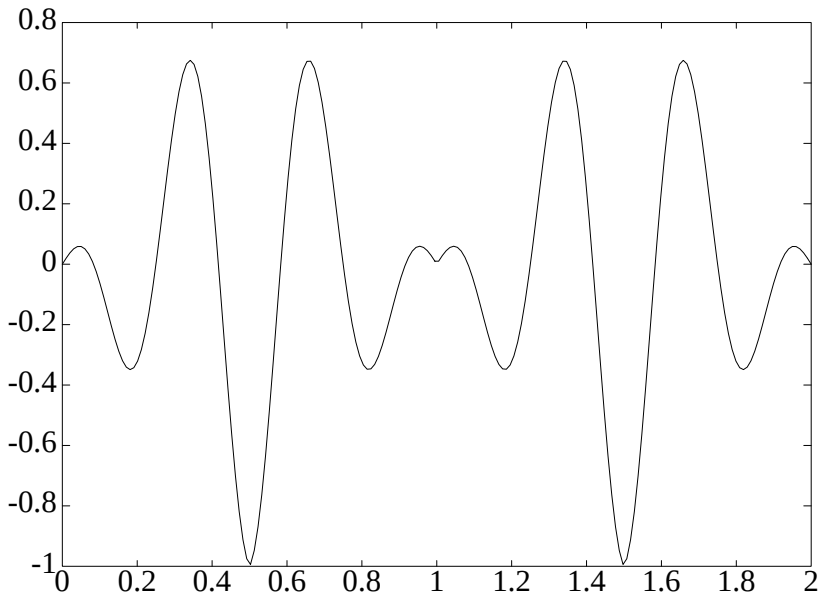


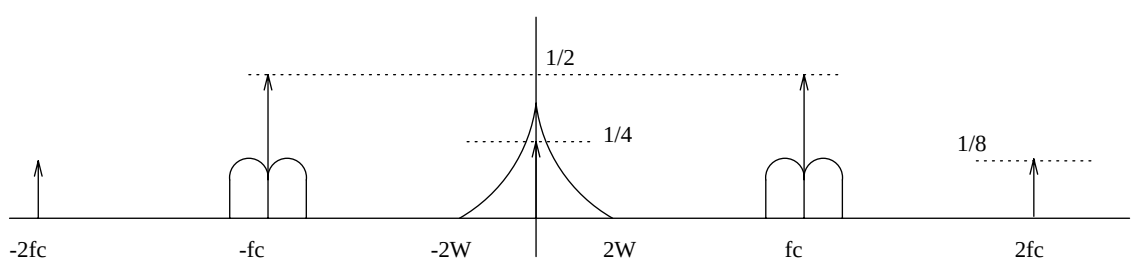
Problem 3.4

$$
\begin{aligned}
y(t) &= x(t) + \frac{1}{2}x^2(t) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2 f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}
$$

Taking the Fourier transform of the previous, we obtain

$$
\begin{aligned}
Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}\left(M(f - f_c) + M(f + f_c)\right) \\
&\quad + \frac{1}{4}\delta(f) + \frac{1}{2}\left(\delta(f - f_c) + \delta(f + f_c)\right) + \frac{1}{8}\left(\delta(f - 2f_c) + \delta(f + 2f_c)\right)
\end{aligned}
$$

The next figure depicts the spectrum $Y(f)$



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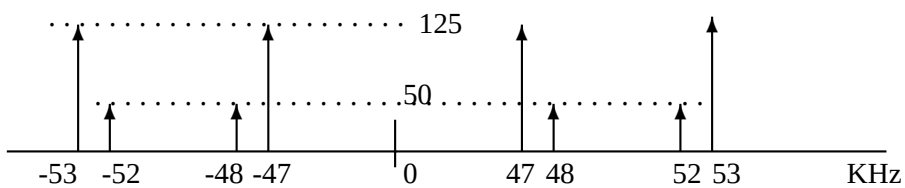
Problem 3.5

$$\begin{aligned} u(t) &= m(t) \cdot c(t) \\ &= 100(2\cos(2\pi 2000t) + 5\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

Thus,

$$\begin{aligned} U(f) &= \frac{100}{2}\left[\delta(f-2000) + \delta(f+2000) + \frac{5}{2}(\delta(f-3000) + \delta(f+3000))\right] \\ &\quad \star[\delta(f-50000) + \delta(f+50000)] \\ &= 50\left[\delta(f-52000) + \delta(f-48000) + \frac{5}{2}\delta(f-53000) + \frac{5}{2}\delta(f-47000)\right. \\ &\quad \left. +\delta(f+52000) + \delta(f+48000) + \frac{5}{2}\delta(f+53000) + \frac{5}{2}\delta(f+47000)\right] \end{aligned}$$

A plot of the spectrum of the modulated signal is given in the next figure



Problem 3.6

The mixed signal $y(t)$ is given by

$$\begin{aligned} y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\ &= \frac{A}{2}m(t)\left[\cos(2\pi 2 f_c t + \theta) + \cos(\theta)\right] \end{aligned}$$

The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2}m(t)\cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{\text{out}} = P_M \frac{A^2}{4}\cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2}P_M$. Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2}\cos^2(\theta)$$

A plot of $\frac{P_{\text{out}}}{P_U}$ for $0 \le \theta \le \pi$ is given in the next figure.

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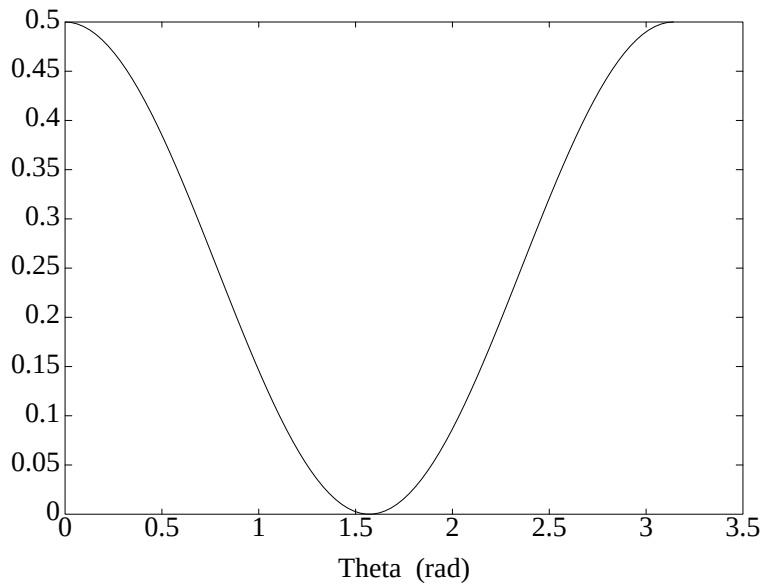
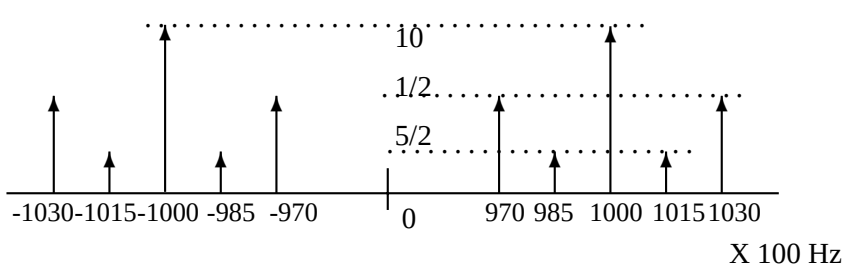


Problem 3.7

1) The spectrum of $u(t)$ is

$$\begin{aligned} U(f) &= \frac{20}{2}[\delta(f-f_c)+\delta(f+f_c)] \\ &+\frac{2}{4}[\delta(f-f_c-1500)+\delta(f-f_c+1500) \\ &+\delta(f+f_c-1500)+\delta(f+f_c+1500)] \\ &+\frac{10}{4}[\delta(f-f_c-3000)+\delta(f-f_c+3000) \\ &+\delta(f+f_c-3000)+\delta(f+f_c+3000)] \end{aligned}$$

The next figure depicts the spectrum of $u(t)$.



2) The square of the modulated signal is

$$\begin{aligned} u^2(t) &= 400\cos^2(2\pi f_c t)+\cos^2(2\pi(f_c-1500)t)+\cos^2(2\pi(f_c+1500)t) \\ &+25\cos^2(2\pi(f_c-3000)t)+25\cos^2(2\pi(f_c+3000)t) \\ &+\text{ terms that are multiples of cosines} \end{aligned}$$

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If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t) \\ &= \cos^2(2\pi 1500t) + \frac{1}{10}\cos(2\pi 1500t) - \frac{1}{2} \end{aligned}$$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. Hence, the modulation index is

$$\alpha = -\frac{201}{400}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &= 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226}$$

Problem 3.8

1)

$$\begin{aligned} u(t) &= m(t)c(t) \\ &= 100(\cos(2\pi 1000t) + 2\cos(2\pi 2000t))\cos(2\pi f_c t) \\ &= 100\cos(2\pi 1000t)\cos(2\pi f_c t) + 200\cos(2\pi 2000t)\cos(2\pi f_c t) \\ &= \frac{100}{2}[\cos(2\pi(f_c + 1000)t) + \cos(2\pi(f_c - 1000)t)] \\ &\quad \frac{200}{2}[\cos(2\pi(f_c + 2000)t) + \cos(2\pi(f_c - 2000)t)] \end{aligned}$$

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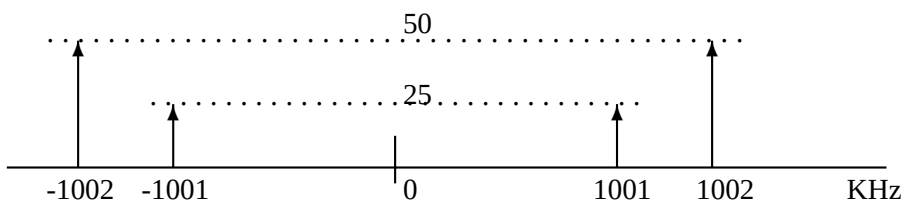
Thus, the upper sideband (USB) signal is

$$u_u(t) = 50\cos(2\pi(f_c + 1000)t) + 100\cos(2\pi(f_c + 2000)t)$$

2) Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U_u(f) &= 25\,(\delta(f - (f_c + 1000)) + \delta(f + (f_c + 1000))) \\ &\quad +50\,(\delta(f - (f_c + 2000)) + \delta(f + (f_c + 2000))) \end{aligned}$$

A plot of $U_u(f)$ is given in the next figure.



Problem 3.9

If we let

$$x(t) = -\Pi\left(\frac{t + \frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t - \frac{T_p}{4}}{\frac{T_p}{2}}\right)$$

then using the results of Problem 2.56, we obtain

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t - nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where

$$\begin{aligned} X(\frac{n}{T_p}) &= \mathcal{F}\left[-\Pi\left(\frac{t + \frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t - \frac{T_p}{4}}{\frac{T_p}{2}}\right)\right]\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(f\frac{T_p}{2})\left(e^{-j2\pi f\frac{T_p}{4}} - e^{j2\pi f\frac{T_p}{4}}\right)\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2}) \end{aligned}$$

Hence, the Fourier transform of $v(t)$ is

$$V(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2})M(f - \frac{n}{T_p})$$

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The bandpass filter will cut-off all the frequencies except the ones centered at $\frac{1}{T_p}$, that is for $n = \pm 1$. Thus, the output spectrum is

$$\begin{aligned} U(f) &= \text{sinc}(\frac{1}{2})(-j)M(f - \frac{1}{T_p}) + \text{sinc}(\frac{1}{2})jM(f + \frac{1}{T_p}) \\ &= -\frac{2}{\pi}jM(f - \frac{1}{T_p}) + \frac{2}{\pi}jM(f + \frac{1}{T_p}) \\ &= \frac{4}{\pi}M(f) \star \left[\frac{1}{2j}\delta(f - \frac{1}{T_p}) - \frac{1}{2j}\delta(f + \frac{1}{T_p})\right] \end{aligned}$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t) = \frac{4}{\pi}m(t)\sin(2\pi\frac{1}{T_p}t)$$

which has the form of a DSB-SC AM signal, with $c(t) = \frac{4}{\pi}\sin(2\pi\frac{1}{T_p}t)$ being the carrier signal.

Problem 3.10

Assume that $s(t)$ is a periodic signal with period T_p, i.e. $s(t) = \sum_n x(t - nT_p)$. Then

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t - nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where $X(\frac{n}{T_p}) = \mathcal{F}[x(t)]|_{f=\frac{n}{T_p}}$. The Fourier transform of $v(t)$ is

$$\begin{aligned} V(f) &= \frac{1}{T_p}\mathcal{F}\left[\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t}\right] \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})M(f - \frac{n}{T_p}) \end{aligned}$$

The bandpass filter will cut-off all the frequency components except the ones centered at $f_c = \pm\frac{1}{T_p}$. Hence, the spectrum at the output of the BPF is

$$U(f) = \frac{1}{T_p}X(\frac{1}{T_p})M(f - \frac{1}{T_p}) + \frac{1}{T_p}X(-\frac{1}{T_p})M(f + \frac{1}{T_p})$$

In the time domain the output of the BPF is given by

$$\begin{aligned} u(t) &= \frac{1}{T_p}X(\frac{1}{T_p})m(t)e^{j2\pi\frac{1}{T_p}t} + \frac{1}{T_p}X^*(\frac{1}{T_p})m(t)e^{-j2\pi\frac{1}{T_p}t} \\ &= \frac{1}{T_p}m(t)\left[X(\frac{1}{T_p})e^{j2\pi\frac{1}{T_p}t} + X^*(\frac{1}{T_p})e^{-j2\pi\frac{1}{T_p}t}\right] \\ &= \frac{1}{T_p}2\text{Re}(\text{X}(\frac{1}{\text{T}_\text{p}}))\text{m(t)}\cos(2\pi\frac{1}{\text{T}_\text{p}}\text{t}) \end{aligned}$$

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As it is observed $u(t)$ has the form a modulated DSB-SC signal. The amplitude of the modulating signal is $A_c = \frac{1}{T_p} 2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))$ and the carrier frequency $f_c = \frac{1}{T_p}$.

Problem 3.11

1) The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]$$

The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)]$$

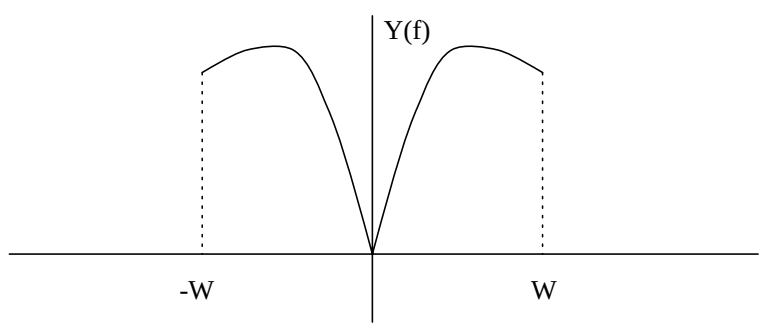
Multiplying the output of the HPF with $A\cos(2\pi(f_c + W)t)$ results in the signal $z(t)$ with spectrum

$$\begin{aligned}
Z(f) &= \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)] \\
&\quad \star \frac{A}{2}[\delta(f - (f_c + W)) + \delta(f + f_c + W)] \\
&= \frac{A^2}{4}(M(f + f_c - f_c - W)u_{-1}(-f + f_c + W - f_c) \\
&\quad + M(f + f_c - f_c + W)u_{-1}(f + f_c + W - f_c) \\
&\quad + M(f - 2f_c - W)u_{-1}(f - 2f_c - W) \\
&\quad + M(f + 2f_c + W)u_{-1}(-f - 2f_c - W)) \\
&= \frac{A^2}{4}(M(f - W)u_{-1}(-f + W) + M(f + W)u_{-1}(f + W) \\
&\quad + M(f - 2f_c - W)u_{-1}(f - 2f_c - W) + M(f + 2f_c + W)u_{-1}(-f - 2f_c - W))
\end{aligned}$$

The LPF will cut-off the double frequency components, leaving the spectrum

$$Y(f) = \frac{A^2}{4}[M(f - W)u_{-1}(-f + W) + M(f + W)u_{-1}(f + W)]$$

The next figure depicts $Y(f)$ for $M(f)$ as shown in Fig. P-3.11.



2) As it is observed from the spectrum $Y(f)$, the system shifts the positive frequency components to the negative frequency axis and the negative frequency components to the positive frequency axis. If we transmit the signal $y(t)$ through the system, then we will get a scaled version of the original spectrum $M(f)$.

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Problem 3.12

The modulated signal can be written as

$$\begin{aligned} u(t) &= m(t)\cos(2\pi f_c t + \phi) \\ &= m(t)\cos(2\pi f_c t)\cos(\phi) - m(t)\sin(2\pi f_c t)\sin(\phi) \\ &= u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t) \end{aligned}$$

where we identify $u_c(t) = m(t)\cos(\phi)$ as the in-phase component and $u_s(t) = m(t)\sin(\phi)$ as the quadrature component. The envelope of the bandpass signal is

$$\begin{aligned} V_u(t) &= \sqrt{u_c^2(t) + u_s^2(t)} = \sqrt{m^2(t)\cos^2(\phi) + m^2(t)\sin^2(\phi)} \\ &= \sqrt{m^2(t)} = |m(t)| \end{aligned}$$

Hence, the envelope is proportional to the absolute value of the message signal.

Problem 3.13

1) The modulated signal is

$$\begin{aligned} u(t) &= 100[1 + m(t)]\cos(2\pi 8 \times 10^5 t) \\ &= 100\cos(2\pi 8 \times 10^5 t) + 100\sin(2\pi 10^3 t)\cos(2\pi 8 \times 10^5 t) \\ &\quad +500\cos(2\pi 2 \times 10^3 t)\cos(2\pi 8 \times 10^5 t) \\ &= 100\cos(2\pi 8 \times 10^5 t) + 50[\sin(2\pi(10^3 + 8 \times 10^5)t) - \sin(2\pi(8 \times 10^5 - 10^3)t)] \\ &\quad +250[\cos(2\pi(2 \times 10^3 + 8 \times 10^5)t) + \cos(2\pi(8 \times 10^5 - 2 \times 10^3)t)] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50[\delta(f - 8 \times 10^5) + \delta(f + 8 \times 10^5)] \\ &\quad +25\left[\frac{1}{j}\delta(f - 8 \times 10^5 - 10^3) - \frac{1}{j}\delta(f + 8 \times 10^5 + 10^3)\right] \\ &\quad -25\left[\frac{1}{j}\delta(f - 8 \times 10^5 + 10^3) - \frac{1}{j}\delta(f + 8 \times 10^5 - 10^3)\right] \\ &\quad +125\left[\delta(f - 8 \times 10^5 - 2 \times 10^3) + \delta(f + 8 \times 10^5 + 2 \times 10^3)\right] \\ &\quad +125\left[\delta(f - 8 \times 10^5 - 2 \times 10^3) + \delta(f + 8 \times 10^5 + 2 \times 10^3)\right] \\ &= 50[\delta(f - 8 \times 10^5) + \delta(f + 8 \times 10^5)] \\ &\quad +25\left[\delta(f - 8 \times 10^5 - 10^3)e^{-j\frac{\pi}{2}} + \delta(f + 8 \times 10^5 + 10^3)e^{j\frac{\pi}{2}}\right] \\ &\quad +25\left[\delta(f - 8 \times 10^5 + 10^3)e^{j\frac{\pi}{2}} + \delta(f + 8 \times 10^5 - 10^3)e^{-j\frac{\pi}{2}}\right] \\ &\quad +125\left[\delta(f - 8 \times 10^5 - 2 \times 10^3) + \delta(f + 8 \times 10^5 + 2 \times 10^3)\right] \\ &\quad +125\left[\delta(f - 8 \times 10^5 - 2 \times 10^3) + \delta(f + 8 \times 10^5 + 2 \times 10^3)\right] \end{aligned}$$

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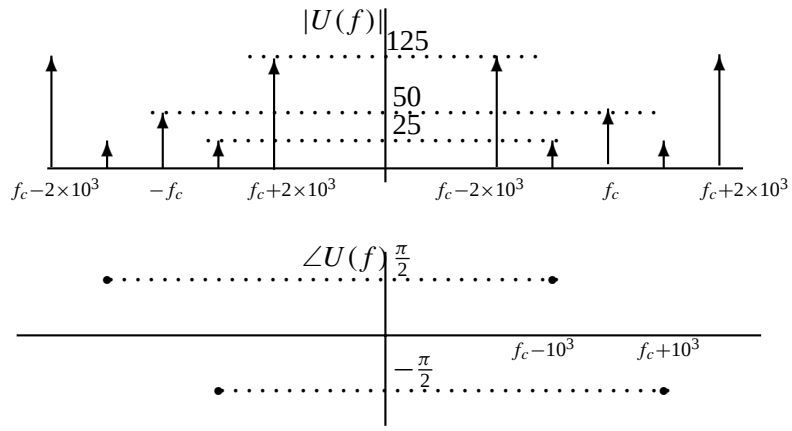

2) The average power in the carrier is

$$P_{\text{carrier}} = \frac{A_c^2}{2} = \frac{100^2}{2} = 5000$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{50^2}{2} + \frac{50^2}{2} + \frac{250^2}{2} + \frac{250^2}{2} = 65000$$

3) The message signal can be written as

$$\begin{aligned} m(t) &= \sin(2\pi 10^3 t) + 5\cos(2\pi 2 \times 10^3 t) \\ &= -10\sin(2\pi 10^3 t) + \sin(2\pi 10^3 t) + 5 \end{aligned}$$

As it is seen the minimum value of $m(t)$ is -6 and is achieved for $\sin(2\pi 10^3 t) = -1$ or $t = \frac{3}{4\times 10^3} + \frac{1}{10^3}k$, with $k \in Z$. Hence, the modulation index is $\alpha = 6$.

4) The power delivered to the load is

$$P_{\text{load}} = \frac{|u(t)|^2}{50} = \frac{100^2(1+m(t))^2\cos^2(2\pi f_c t)}{50}$$

The maximum absolute value of $1+m(t)$ is 6.025 and is achieved for $\sin(2\pi 10^3 t) = \frac{1}{20}$ or $t = \frac{\arcsin(\frac{1}{20})}{2\pi 10^3} + \frac{k}{10^3}$. Since $2 \times 10^3 \ll f_c$ the peak power delivered to the load is approximately equal to

$$\max(P_{\text{load}}) = \frac{(100 \times 6.025)^2}{50} = 72.6012$$

Problem 3.14

1)

$$\begin{aligned} u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\ &= 20(1 + \frac{1}{2}\cos(200\pi t))\cos(2000\pi t) \end{aligned}$$

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The modulating signal is $m(t) = \cos(2\pi 100t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000t)$.

2) Since $-1 \leq \cos(2\pi 100t) \leq 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

Problem 3.15

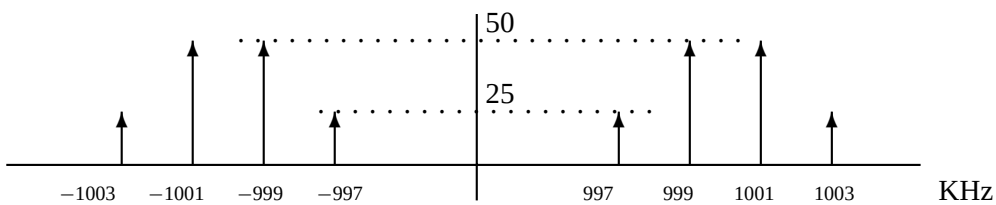
1) The modulated signal is written as

$$\begin{aligned} u(t) &= 100(2\cos(2\pi 10^3 t) + \cos(2\pi 3\times 10^3 t))\cos(2\pi f_c t) \\ &= 200\cos(2\pi 10^3 t)\cos(2\pi f_c t) + 100\cos(2\pi 3\times 10^3 t)\cos(2\pi f_c t) \\ &= 100\left[\cos(2\pi(f_c + 10^3)t) + \cos(2\pi(f_c - 10^3)t)\right] \\ &\quad +50\left[\cos(2\pi(f_c + 3\times 10^3)t) + \cos(2\pi(f_c - 3\times 10^3)t)\right] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50\left[\delta(f - (f_c + 10^3)) + \delta(f + f_c + 10^3)\right. \\ &+ \left.\delta(f - (f_c - 10^3)) + \delta(f + f_c - 10^3)\right] \\ &+ 25\left[\delta(f - (f_c + 3\times 10^3)) + \delta(f + f_c + 3\times 10^3)\right. \\ &+ \left.\delta(f - (f_c - 3\times 10^3)) + \delta(f + f_c - 3\times 10^3)\right] \end{aligned}$$

The spectrum of the signal is depicted in the next figure



2) The average power in the frequencies $f_c + 1000$ and $f_c - 1000$ is

$$P_{f_c+1000} = P_{f_c-1000} = \frac{100^2}{2} = 5000$$

The average power in the frequencies $f_c + 3000$ and $f_c - 3000$ is

$$P_{f_c+3000} = P_{f_c-3000} = \frac{50^2}{2} = 1250$$

Problem 3.16

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1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of$\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100, m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned}
u_l(t) &= 100\,[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)]\cos(2\pi f_c t) \\
&+ 100\,[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)]\sin(2\pi f_c t) \\
&= 100\,[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)] \\
&+ 200\,[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)] \\
&= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t)
\end{aligned}$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned}
U_l(f) &= 50\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\
&+ 100j\,(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)) \\
&= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000)
\end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned}
|U_l(f)| &= \sqrt{50^2 + 100^2}\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\
&= 10\sqrt{125}\,(\delta(f - f_c + 1000) + \delta(f + f_c - 1000))
\end{aligned}$$

Problem 3.17

The input to the upper LPF is

$$\begin{aligned}
u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\
&= \frac{1}{2}[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)]
\end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned}
u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\
&= \frac{1}{2}[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)]
\end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned}
y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\
y_l(t) &= \sin(2\pi(f_1 - f_m)t)
\end{aligned}$$

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The output of the Weaver's modulator is

$$u(t) = \cos(2\pi(f_1 - f_m)t)\cos(2\pi f_2 t) - \sin(2\pi(f_1 - f_m)t)\sin(2\pi f_2 t)$$

which has the form of a SSB signal since $\sin(2\pi(f_1 - f_m)t)$ is the Hilbert transform of $\cos(2\pi(f_1 - f_m)t)$. If we write $u(t)$ as

$$u(t) = \cos(2\pi(f_1 + f_2 - f_m)t)$$

then with $f_1+f_2-f_m = f_c+f_m$ we obtain an USSB signal centered at f_c, whereas with $f_1+f_2-f_m = f_c-f_m$ we obtain the LSSB signal. In both cases the choice of f_c and f_1 uniquely determine f_2.

Problem 3.18

The signal $x(t)$ is $m(t)+\cos(2\pi f_0 t)$. The spectrum of this signal is $X(f) = M(f)+\frac{1}{2}(\delta(f-f_0)+\delta(f+f_0))$ and its bandwidth equals to $W_x = f_0$. The signal $y_1(t)$ after the Square Law Device is

$$\begin{aligned} y_1(t) &= x^2(t) = (m(t) + \cos(2\pi f_0 t))^2 \\ &= m^2(t) + \cos^2(2\pi f_0 t) + 2m(t)\cos(2\pi f_0 t) \\ &= m^2(t) + \frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_0 t) + 2m(t)\cos(2\pi f_0 t) \end{aligned}$$

The spectrum of this signal is given by

$$Y_1(f) = M(f) \star M(f) + \frac{1}{2}\delta(f) + \frac{1}{4}(\delta(f - 2f_0) + \delta(f + 2f_0)) + M(f - f_0) + M(f + f_0)$$

and its bandwidth is $W_1 = 2f_0$. The bandpass filter will cut-off the low-frequency components $M(f) \star M(f) + \frac{1}{2}\delta(f)$ and the terms with the double frequency components $\frac{1}{4}(\delta(f-2f_0)+\delta(f+2f_0))$. Thus the spectrum $Y_2(f)$ is given by

$$Y_2(f) = M(f - f_0) + M(f + f_0)$$

and the bandwidth of $y_2(t)$ is $W_2 = 2W$. The signal $y_3(t)$ is

$$y_3(t) = 2m(t)\cos^2(2\pi f_0 t) = m(t) + m(t)\cos(2\pi f_0 t)$$

with spectrum

$$Y_3(t) = M(f) + \frac{1}{2}(M(f - f_0) + M(f + f_0))$$

and bandwidth $W_3 = f_0 + W$. The lowpass filter will eliminate the spectral components $\frac{1}{2}(M(f - f_0) + M(f + f_0))$, so that $y_4(t) = m(t)$ with spectrum $Y_4 = M(f)$ and bandwidth $W_4 = W$. The next figure depicts the spectra of the signals $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$.

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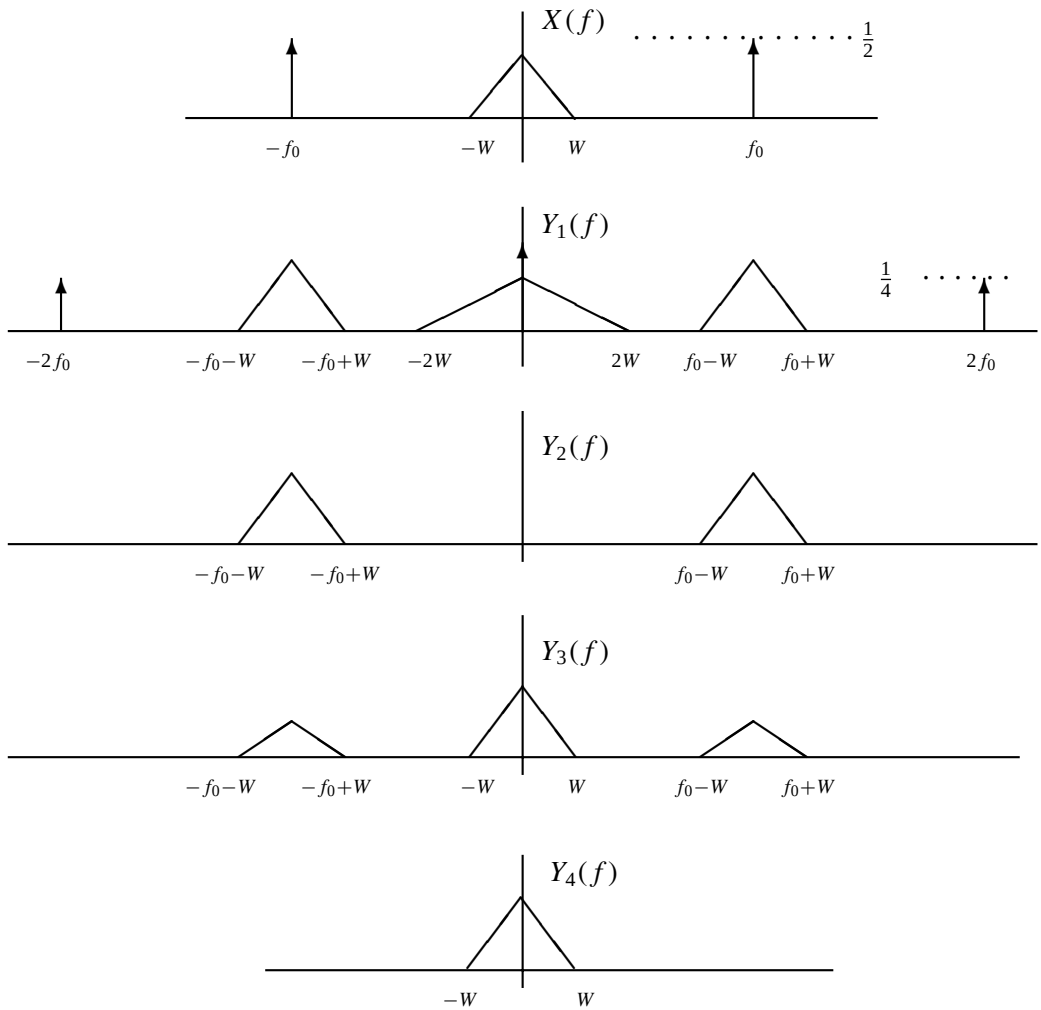

Problem 3.19

1)

$$\begin{aligned} y(t) &= ax(t) + bx^2(t) \\ &= a(m(t) + \cos(2\pi f_0 t)) + b(m(t) + \cos(2\pi f_0 t))^2 \\ &= am(t) + bm^2(t) + a\cos(2\pi f_0 t) \\ &\quad + b\cos^2(2\pi f_0 t) + 2bm(t)\cos(2\pi f_0 t) \end{aligned}$$

2) The filter should reject the low frequency components, the terms of double frequency and pass only the signal with spectrum centered at f_0. Thus the filter should be a BPF with center frequency f_0 and bandwidth W such that $f_0 - W_M > f_0 - \frac{W}{2} > 2W_M$ where W_M is the bandwidth of the message signal $m(t)$.

3) The AM output signal can be written as

$$u(t) = a(1 + \frac{2b}{a}m(t))\cos(2\pi f_0 t)$$

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Since $A_m = \max[|m(t)|]$ we conclude that the modulation index is

$$\alpha = \frac{2bA_m}{a}$$

Problem 3.20

1) When USSB is employed the bandwidth of the modulated signal is the same with the bandwidth of the message signal. Hence,

$$W_{\text{USSB}} = W = 10^4 \text{ Hz}$$

2) When DSB is used, then the bandwidth of the transmitted signal is twice the bandwidth of the message signal. Thus,

$$W_{\text{DSB}} = 2W = 2 \times 10^4 \text{ Hz}$$

3) If conventional AM is employed, then

$$W_{\text{AM}} = 2W = 2 \times 10^4 \text{ Hz}$$

4) Using Carson's rule, the effective bandwidth of the FM modulated signal is

$$B_c = (2\beta + 1)W = 2\left(\frac{k_f \max[|m(t)|]}{W} + 1\right)W = 2(k_f + W) = 140000 \text{ Hz}$$

Problem 3.21

1) The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u_{-1}(f + f_c)H(f + f_c) = 2\begin{cases} \frac{1}{W}f + \frac{1}{2} & |f| \leq \frac{W}{2} \\ 1 & \frac{W}{2} < f \leq W \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
h_l(t) &= \mathcal{F}^{-1}[H_l(f)] = \int_{-\frac{W}{2}}^{W} H_l(f)e^{j2\pi ft}df \\
&= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} (\frac{1}{W}f + \frac{1}{2})e^{j2\pi ft}df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi ft}df \\
&= \frac{2}{W}\left(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2t^2}e^{j2\pi ft}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t}e^{j2\pi ft}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t}e^{j2\pi ft}\Bigg|_{\frac{W}{2}}^{W} \\
&= \frac{1}{j\pi t}e^{j2\pi Wt} + \frac{j}{\pi^2t^2W}\sin(\pi Wt) \\
&= \frac{j}{\pi t}\left[\text{sinc}(Wt) - e^{j2\pi Wt}\right]
\end{aligned}$$

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2) An expression for the modulated signal is obtained as follows

$$\begin{aligned} u(t) &= \mathrm{Re}[(m(t) \star h_l(t))e^{j2\pi f_c t}] \\ &= \mathrm{Re}\left[(m(t) \star \frac{j}{\pi t}(\mathrm{sinc}(Wt) - e^{j2\pi Wt}))e^{j2\pi f_c t}\right] \\ &= \mathrm{Re}\left[(m(t) \star (\frac{j}{\pi t}\mathrm{sinc}(Wt)))e^{j2\pi f_c t} + (m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt})e^{j2\pi f_c t}\right] \end{aligned}$$

Note that

$$\mathcal{F}[m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt}] = -M(f)\mathrm{sgn}(f - W) = M(f)$$

since $\mathrm{sgn}(f - W) = -1$ for $f < W$. Thus,

$$\begin{aligned} u(t) &= \mathrm{Re}\left[(m(t) \star (\frac{j}{\pi t}\mathrm{sinc}(Wt)))e^{j2\pi f_c t} + m(t)e^{j2\pi f_c t}\right] \\ &= m(t)\cos(2\pi f_c t) - m(t) \star (\frac{1}{\pi t}\mathrm{sinc}(Wt))\sin(2\pi f_c t) \end{aligned}$$

Problem 3.22

a) A DSB modulated signal is written as

$$\begin{aligned} u(t) &= Am(t)\cos(2\pi f_0 t + \phi) \\ &= Am(t)\cos(\phi)\cos(2\pi f_0 t) - Am(t)\sin(\phi)\sin(2\pi f_0 t) \end{aligned}$$

Hence,

$$\begin{aligned} x_c(t) &= Am(t)\cos(\phi) \\ x_s(t) &= Am(t)\sin(\phi) \\ V(t) &= \sqrt{A^2 m^2(t)(\cos^2(\phi) + \sin^2(\phi))} = |Am(t)| \\ \Theta(t) &= \arctan\left(\frac{Am(t)\cos(\phi)}{Am(t)\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi \end{aligned}$$

b) A SSB signal has the form

$$u_{\mathrm{SSB}}(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Thus, for the USSB signal (minus sign)

$$\begin{aligned} x_c(t) &= Am(t) \\ x_s(t) &= A\hat{m}(t) \\ V(t) &= \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)} \\ \Theta(t) &= \arctan\left(\frac{\hat{m}(t)}{m(t)}\right) \end{aligned}$$

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For the LSSB signal (plus sign)

$$\begin{aligned}
x_c(t) &= Am(t) \\
x_s(t) &= -A\hat{m}(t) \\
V(t) &= \sqrt{A^2(m^2(t)+\hat{m}^2(t))} = A\sqrt{m^2(t)+\hat{m}^2(t)} \\
\Theta(t) &= \arctan\left(-\frac{\hat{m}(t)}{m(t)}\right)
\end{aligned}$$

c) If conventional AM is employed, then

$$\begin{aligned}
u(t) &= A(1+m(t))\cos(2\pi f_0 t+\phi) \\
&= A(1+m(t))\cos(\phi)\cos(2\pi f_0 t) - A(1+m(t))\sin(\phi)\sin(2\pi f_0 t)
\end{aligned}$$

Hence,

$$\begin{aligned}
x_c(t) &= A(1+m(t))\cos(\phi) \\
x_s(t) &= A(1+m(t))\sin(\phi) \\
V(t) &= \sqrt{A^2(1+m(t))^2(\cos^2(\phi)+\sin^2(\phi))} = A|(1+m(t))| \\
\Theta(t) &= \arctan\left(\frac{A(1+m(t))\cos(\phi)}{A(1+m(t))\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi
\end{aligned}$$

Problem 3.23

1) If SSB is employed, the transmitted signal is

$$u(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Provided that the spectrum of $m(t)$ does not contain any impulses at the origin $P_M = P_{\hat{M}} = \frac{1}{2}$ and

$$P_{\text{SSB}} = \frac{A^2 P_M}{2} + \frac{A^2 P_{\hat{M}}}{2} = A^2 P_M = 400\frac{1}{2} = 200$$

The bandwidth of the modulated signal $u(t)$ is the same with that of the message signal. Hence,

$$W_{\text{SSB}} = 10000 \text{ Hz}$$

2) In the case of DSB-SC modulation $u(t) = Am(t)\cos(2\pi f_0 t)$. The power content of the modulated signal is

$$P_{\text{DSB}} = \frac{A^2 P_M}{2} = 200\frac{1}{2} = 100$$

and the bandwidth $W_{\text{DSB}} = 2W = 20000$ Hz.

3) If conventional AM is employed with modulation index $\alpha = 0.6$, the transmitted signal is

$$u(t) = A[1+\alpha m(t)]\cos(2\pi f_0 t)$$

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The power content is

$$P_{\text{AM}} = \frac{A^2}{2} + \frac{A^2\alpha^2 P_M}{2} = 200 + 200 \cdot 0.6^2 \cdot 0.5 = 236$$

The bandwidth of the signal is $W_{\text{AM}} = 2W = 20000$ Hz.

4) If the modulation is FM with $k_f = 50000$, then

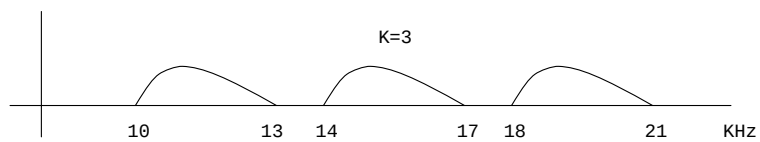
$$P_{\text{FM}} = \frac{A^2}{2} = 200$$

and the effective bandwidth is approximated by Carson's rule as

$$B_c = 2(\beta + 1)W = 2\left(\frac{50000}{W} + 1\right)W = 120000 \text{ Hz}$$

Problem 3.24

1) The next figure illustrates the spectrum of the SSB signal assuming that USSB is employed and $K = 3$. Note, that only the spectrum for the positive frequencies has been plotted.



2) With $LK = 60$ the possible values of the pair (L, K) (or (K, L)) are $\{(1, 60), (2, 30), (3, 20), (4, 15), (6, 10)\}$. As it is seen the minimum value of $L + K$ is achieved for $L = 6$, $K = 10$ (or $L = 10$, $K = 6$).

3) Assuming that $L = 6$ and $K = 10$ we need 16 carriers with frequencies

$$\begin{aligned} f_{k_1} &= 10 \text{ KHz} & f_{k_2} &= 14 \text{ KHz} \\ f_{k_3} &= 18 \text{ KHz} & f_{k_4} &= 22 \text{ KHz} \\ f_{k_5} &= 26 \text{ KHz} & f_{k_6} &= 30 \text{ KHz} \\ f_{k_7} &= 34 \text{ KHz} & f_{k_8} &= 38 \text{ KHz} \\ f_{k_9} &= 42 \text{ KHz} & f_{k_{10}} &= 46 \text{ KHz} \end{aligned}$$

and

$$\begin{aligned} f_{l_1} &= 290 \text{ KHz} & f_{l_2} &= 330 \text{ KHz} \\ f_{l_3} &= 370 \text{ KHz} & f_{l_4} &= 410 \text{ KHz} \\ f_{l_5} &= 450 \text{ KHz} & f_{l_6} &= 490 \text{ KHz} \end{aligned}$$

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