Solution to Chapter 6 Problems

Problem 6.1

The spectrum of the signal at the output of the LPF is $\mathcal{S}_{s,o}(f) = \mathcal{S}_s(f)|\Pi(\frac{f}{2W})|^2$. Hence, the signal power is

$$\begin{aligned} P_{s,o} &= \int_{-\infty}^{\infty} \mathcal{S}_{s,o}(f)df = \int_{-W}^{W} \frac{P_0}{1+(f/B)^2}df \\ &= P_0 B \arctan(\frac{f}{B})\Big|_{-W}^{W} = 2P_0 B \arctan(\frac{W}{B}) \end{aligned}$$

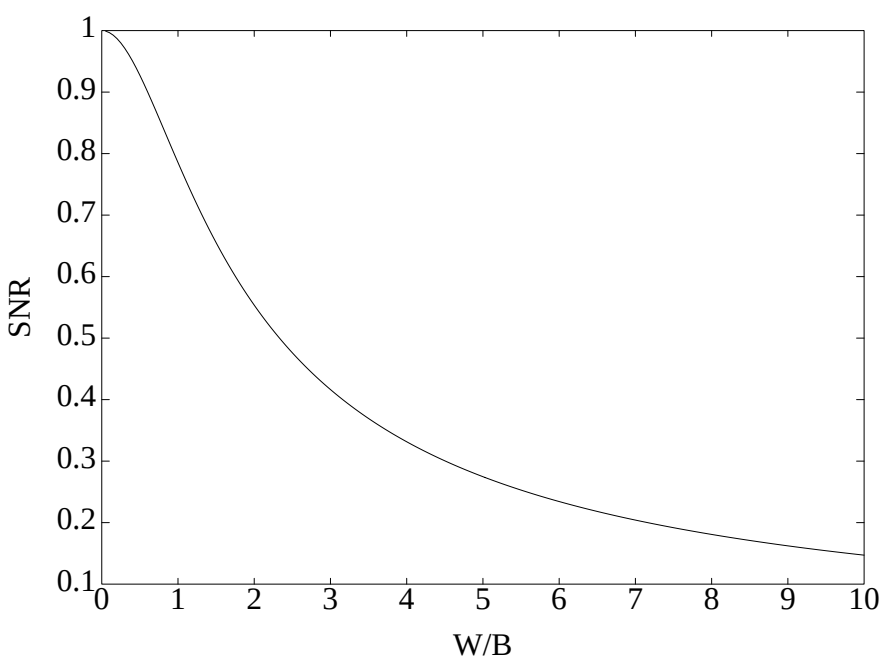
Similarly, noise power at the output of the lowpass filter is

$$P_{n,o} = \int_{-W}^{W} \frac{N_0}{2}df = N_0 W$$

Thus, the SNR is given by

$$\text{SNR} = \frac{2P_0 B \arctan(\frac{W}{B})}{N_0 W} = \frac{2P_0}{N_0}\frac{\arctan(\frac{W}{B})}{\frac{W}{B}}$$

In the next figure we plot SNR as a function of $\frac{W}{B}$ and for $\frac{2P_0}{N_0} = 1$.



Problem 6.2

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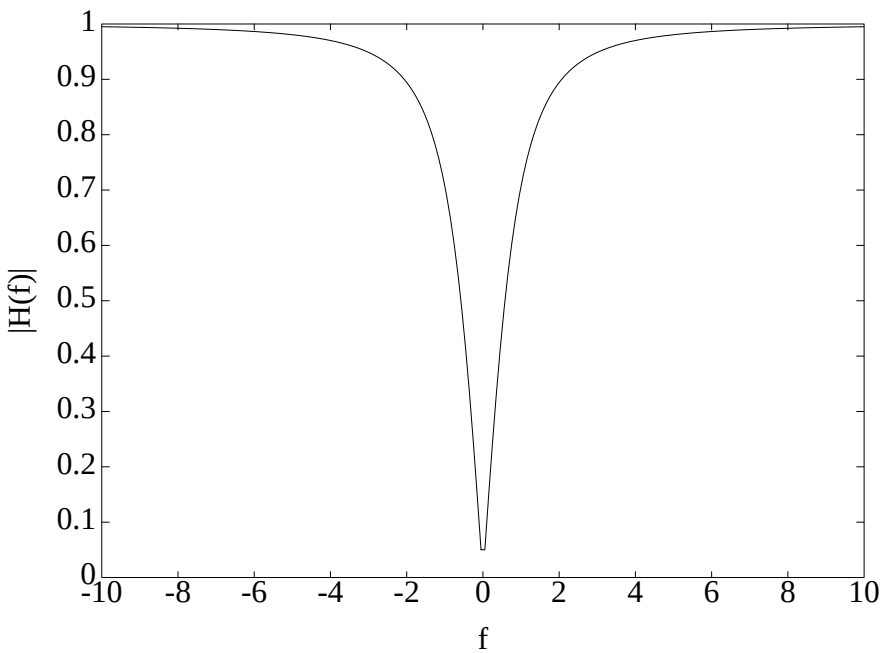
1) The transfer function of the RC filter is

$$H(s) = \frac{R}{\frac{1}{Cs} + R} = \frac{RCs}{1 + RCs}$$

with $s = j2\pi f$. Hence, the magnitude frequency response is

$$|H(f)| = \left(\frac{4\pi^2 (RC)^2 f^2}{1 + 4\pi^2 (RC)^2 f^2} \right)^{\frac{1}{2}}$$

This function is plotted in the next figure for f in $[-10, 10]$ and $4\pi^2 (RC)^2 = 1$.



2) The overall system is the cascade of the RC and the LPF filter. If the bandwidth of the LPF is W, then the transfer function of the system is

$$V(f) = \frac{j2\pi RCf}{1 + j2\pi RCf} \Pi(\frac{f}{2W})$$

The next figure depicts $|V(f)|$ for $W = 5$ and $4\pi^2 (RC)^2 = 1$.

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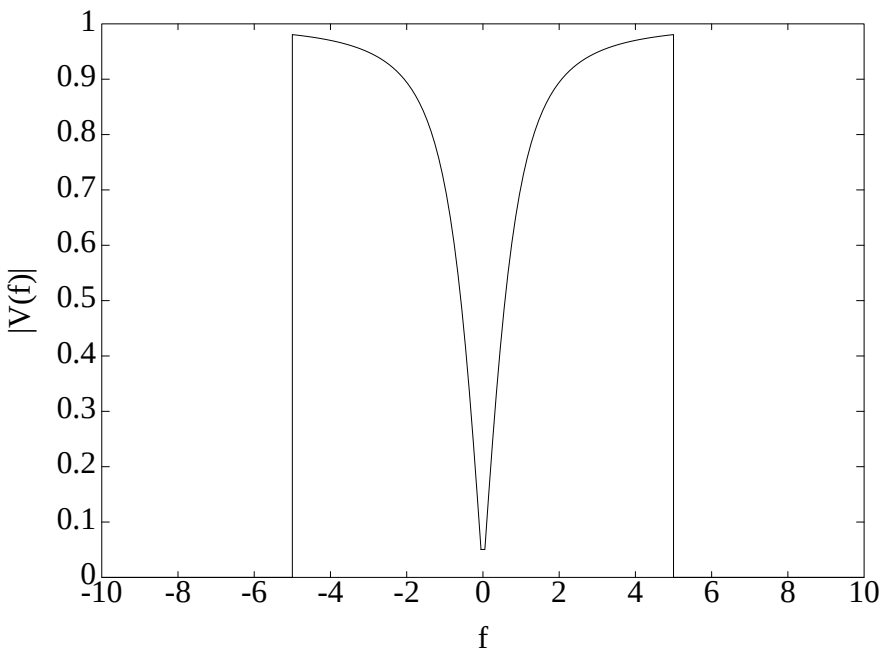

3) The noise output power is

$$\begin{aligned} P_n &= \int_{-W}^{W} \frac{4\pi^2(RC)^2 f^2}{1+4\pi^2(RC)^2 f^2}\frac{N_0}{2}df \\ &= N_0 W - \frac{N_0}{2}\int_{-W}^{W}\frac{1}{1+4\pi^2(RC)^2 f^2}df \\ &= N_0 W - \frac{N_0}{2}\frac{1}{2\pi RC}\arctan(2\pi RCf)\Big|_{-W}^{W} \\ &= N_0 W - \frac{N_0}{2\pi RC}\arctan(2\pi RCW) \end{aligned}$$

The output signal is a sinusoidal with frequency f_c and amplitude $A|V(f_c)|$. Since $f_c < W$ we conclude that the amplitude of the sinusoidal ouput signal is

$$A|H(f_c)| = A\sqrt{\frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}}$$

and the output signal power

$$P_s = \frac{A^2}{2}\frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}$$

Thus, the SNR at the ouput of the LPF is

$$\text{SNR} = \frac{\frac{A^2}{2}\frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{N_0 W - \frac{N_0}{2\pi RC}\arctan(2\pi RCW)} = \frac{\frac{A^2}{N_0}\frac{\pi RC f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{2\pi RCW - \arctan(2\pi RCW)}$$

In the next figure we plot

$$G(W) = \frac{1}{2\pi RCW - \arctan(2\pi RCW)}$$

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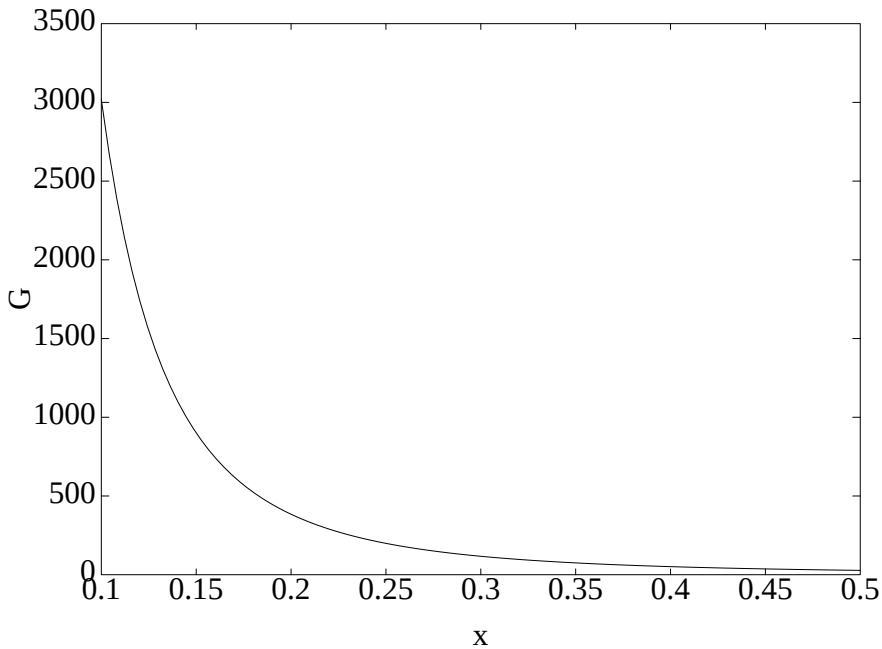
as a function of $x = 2\pi RCW$, when the latter varies from 0.1 to 0.5.



Problem 6.3

The noise power content of the received signal $r(t) = u(t) + n(t)$ is

$$P_n = \int_{-\infty}^{\infty} \mathcal{S}_n(f)df = \frac{N_0}{2} \times 4W = 2N_0W$$

If we write $n(t)$ as

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

then,

$$\begin{aligned} n(t)\cos(2\pi f_c t) &= n_c(t)\cos^2(2\pi f_c t) - n_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}n_c(t) + \frac{1}{2}n_c(t)\cos(2\pi 2 f_c t) - n_s(t)\sin(2\pi 2 f_c t) \end{aligned}$$

The noise signal at the output of the LPF is $\frac{1}{2}n_c(t)$ with power content

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n = \frac{N_0W}{2}$$

If the DSB modulated signal is $u(t) = m(t)\cos(2\pi f_c t)$, then its autocorrelation function is $\bar{R}_u(\tau) = \frac{1}{2}R_M(\tau)\cos(2\pi f_c \tau)$ and its power

$$P_u = \bar{R}_u(0) = \frac{1}{2}R_M(0) = \int_{-\infty}^{\infty} \mathcal{S}_u(f)df = 2WP_0$$

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From this relation we find $R_M(0) = 4WP_0$. The signal at the output of the LPF is $y(t) = \frac{1}{2}m(t)$ with power content

$$P_{s,o} = \frac{1}{4}E[m^2(t)] = \frac{1}{4}R_M(0) = WP_0$$

Hence, the SNR at the output of the demodulator is

$$\text{SNR} = \frac{P_{s,o}}{P_{n,o}} = \frac{WP_0}{\frac{N_0 W}{2}} = \frac{2P_0}{N_0}$$

Problem 6.4

First we determine the baseband signal to noise ratio $(\frac{S}{N})_b$. With $W = 1.5 \times 10^6$, we obtain

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = \frac{P_R}{2 \times 0.5 \times 10^{-14} \times 1.5 \times 10^6} = \frac{P_R 10^8}{1.5}$$

Since the channel attenuation is 90 db, then

$$10 \log \frac{P_T}{P_R} = 90 \Longrightarrow P_R = 10^{-9} P_T$$

Hence,

$$\left(\frac{S}{N}\right)_b = \frac{P_R 10^8}{1.5} = \frac{10^8 \times 10^{-9} P_T}{1.5} = \frac{P_T}{15}$$

1) If USSB is employed, then

$$\left(\frac{S}{N}\right)_{o,\text{USSB}} = \left(\frac{S}{N}\right)_b = 10^3 \Longrightarrow P_T = 15 \times 10^3 = 15 \text{ KWatts}$$

2) If conventional AM is used, then

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \eta \left(\frac{S}{N}\right)_b = \eta \frac{P_T}{15}$$

where, $\eta = \frac{\alpha^2 P_{M_n}}{1+\alpha^2 P_{M_n}}$. Since, $\max[|m(t)| = 1$, we have

$$P_{M_n} = P_M = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{3}$$

and, therefore

$$\eta = \frac{0.25 \times \frac{1}{3}}{1 + 0.25 \times \frac{1}{3}} = \frac{1}{13}$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{1}{13}\frac{P_T}{15} = 10^3 \Longrightarrow P_T = 195 \text{ KWatts}$$

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3) For DSB modulation

$$\left(\frac{S}{N}\right)_{o,\text{DSB}} = \left(\frac{S}{N}\right)_b = \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 15 \text{ KWatts}$$

Problem 6.5

1) Since $|H(f)| = 1$ for $f = |f_c \pm f_m|$, the signal at the output of the noise-limiting filter is

$$r(t) = 10^{-3}[1 + \alpha\cos(2\pi f_m t + \phi)]\cos(2\pi f_c t) + n(t)$$

The signal power is

$$\begin{aligned} P_R &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} 10^{-6}[1 + \alpha\cos(2\pi f_m t + \phi)]^2\cos^2(2\pi f_c t)dt \\ &= \frac{10^{-6}}{2}[1 + \frac{\alpha^2}{2}] = 56.25 \times 10^{-6} \end{aligned}$$

The noise power at the output of the noise-limiting filter is

$$P_{n,o} = \frac{1}{2}P_{n_c} = \frac{1}{2}P_n = \frac{1}{2}\frac{N_0}{2} \times 2 \times 2500 = 25 \times 10^{-10}$$

2) Multiplication of $r(t)$ by $2\cos(2\pi f_c t)$ yields

$$\begin{aligned} y(t) &= \frac{10^{-3}}{2}[1 + \alpha\cos(2\pi f_m t)]2 + \frac{1}{2}n_c(t)2 \\ &\quad + \text{double frequency terms} \end{aligned}$$

The LPF rejects the double frequency components and therefore, the output of the filter is

$$v(t) = 10^{-3}[1 + \alpha\cos(2\pi f_m t)] + n_c(t)$$

If the dc component is blocked, then the signal power at the output of the LPF is

$$P_o = \frac{10^{-6}}{2}0.5^2 = 0.125 \times 10^{-6}$$

whereas, the output noise power is

$$P_{n,o} = P_{n_c} = P_n = 2\frac{N_0}{2}2000 = 40 \times 10^{-10}$$

where we have used the fact that the lowpass filter has a bandwidth of 1000 Hz. Hence, the output SNR is

$$\text{SNR} = \frac{0.125 \times 10^{-6}}{40 \times 10^{-10}} = 31.25 \quad 14.95 \text{ db}$$

Problem 6.6

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The one-sided noise equivalent bandwidth is defined as

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{\max}}$$

It is usually convenient to substitute $|H(f)|^2_{f=0}$ for $|H(f)|^2_{\max}$ in the denominator, since the peaking of the magnitude transfer function may be high (especially for small ζ) creating in this way anomalies. On the other hand if ζ is less, but close, to one, $|H(f)|^2_{\max}$ can be very well approximated by $|H(f)|^2_{f=0}$. Hence,

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{f=0}}$$

and since

$$|H(f)|^2 = \frac{\omega_n^2 + j2\pi f\left(2\zeta\omega_n - \frac{\omega_n^2}{K}\right)}{\omega_n^2 - 4\pi^2 f^2 + j2\pi f 2\zeta\omega_n}$$

we find that $|H(0)| = 1$. Therefore,

$$B_{eq} = \int_0^\infty |H(f)|^2 df$$

For the passive second order filter

$$H(s) = \frac{s(2\zeta\omega_n - \frac{\omega_n^2}{K}) + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$\tau_1 \gg 1$, so that $\frac{\omega_n^2}{K} = \frac{1}{\tau_1} \approx 0$ and

$$H(s) = \frac{s2\zeta\omega_n + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

The B_{eq} can be written as

$$B_{eq} = \frac{1}{4\pi j}\int_{-j\infty}^{j\infty} H(s)H(-s)ds$$

Since, $H(s) = \frac{KG(s)/s}{1+KG(s)/s}$ we obtain $\lim_{|s|\to\infty} H(s)H(-s) = 0$. Hence, the integral for B_{eq} can be taken along a contour, which contains the imaginary axis and the left half plane. Furthermore, since $G(s)$ is a rational function of s, the integral is equal to half the sum of the residues of the left half plane poles of $H(s)H(-s)$. Hence,

$$\begin{aligned}
B_{eq} &= \frac{1}{2}\Bigg[(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}} \\
&\quad +(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}\Bigg] \\
&= \frac{\omega_n}{8}(4\zeta + \frac{1}{\zeta}) = \frac{1+4\zeta^2}{8\zeta/\omega_n} \\
&= \frac{1+\omega_n^2\tau_2^2 + (\frac{\omega_n}{K})^2 + 2\frac{\omega_n^2\tau_2}{K}}{8\zeta/\omega_n} \\
&\approx \frac{1+\omega_n^2\tau_2^2}{8\zeta/\omega_n}
\end{aligned}$$

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where we have used the approximation $\frac{\omega_n}{K} \approx 0$.

Problem 6.7

1) In the case of DSB, the output of the receiver noise-limiting filter is

$$\begin{aligned} r(t) &= u(t) + n(t) \\ &= A_c m(t)\cos(2\pi f_c t + \phi_c(t)) \\ &\quad + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \end{aligned}$$

The power of the received signal is $P_s = \frac{A_c^2}{2} P_m$, whereas the power of the noise

$$P_{n,o} = \frac{1}{2} P_{n_c} + \frac{1}{2} P_{n_s} = P_n$$

Hence, the SNR at the output of the noise-limiting filter is

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2 P_m}{2P_n}$$

Assuming coherent demodulation, the output of the demodulator is

$$y(t) = \frac{1}{2}[A_c m(t) + n_c]$$

The output signal power is $P_o = \frac{1}{4} A_c^2 P_m$ whereas the output noise power

$$P_{n,o} = \frac{1}{4} P_{n_c} = \frac{1}{4} P_n$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{A_c^2 P_m}{P_n}$$

and the demodulation gain is given by

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = 2$$

2) In the case of SSB, the output of the receiver noise-limiting filter is

$$r(t) = A_c m(t)\cos(2\pi f_c t) \pm A_c \hat{m}(t)\sin(2\pi f_c t) + n(t)$$

The received signal power is $P_s = A_c^2 P_m$, whereas the received noise power is $P_{n,o} = P_n$. At the output of the demodulator

$$y(t) = \frac{A_c}{2} m(t) + \frac{1}{2} n_c(t)$$

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with $P_o = \frac{1}{4}A_c^2 P_m$ and $P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$. Therefore,

$$\text{dem. gain } = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\frac{A_c^2 P_m}{P_n}}{\frac{A_c^2 P_m}{P_n}} = 1$$

3) In the case of conventional AM modulation, the output of the receiver noise-limiting filter is

$$r(t) = [A_c(1 + \alpha m_n(t)) + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The total pre-detection power in the signal is

$$P_s = \frac{A_c^2}{2}(1 + \alpha^2 P_{M_n})$$

In this case, the demodulation gain is given by

$$\text{dem. gain } = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{2\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}$$

The highest gain is achieved for $\alpha = 1$, that is 100% modulation.

4) For an FM system, the output of the receiver front-end (bandwidth B_c) is

$$\begin{aligned} r(t) &= A_c \cos(2\pi f_c t + \phi(t)) + n(t) \\ &= A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) + n(t) \end{aligned}$$

The total signal input power is $P_{s,\text{lim}} = \frac{A_c^2}{2}$, whereas the pre-detection noise power is

$$P_{n,\text{lim}} = \frac{N_0}{2}2B_c = N_0 B_c = N_0 2(\beta_f + 1)W$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_f + 1)W}$$

The output (post-detection) signal to noise ratio is

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{3k_f^2 A_c^2 P_M}{2N_0 W^3}$$

Thus, the demodulation gain is

$$\text{dem. gain } = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{3\beta_f^2 P_M 2(\beta_f + 1)}{(\max[|m(t)|])^2} = 6\beta_f^2(\beta_f + 1)P_{M_n}$$

5) Similarly for the PM case we find that

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_p + 1)W}$$

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and

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{k_p^2 A_c^2 P_M}{2N_0 W}$$

Thus, the demodulation gain for a PM system is

$$\text{dem. gain } = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\beta_p^2 P_M 2(\beta_p + 1)}{(\max[|m(t)|])^2} = 2\beta_p^2(\beta_p + 1)P_{M_n}$$

Problem 6.8

1) Since the channel attenuation is 80 db, then

$$10\log\frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8}P_T = 10^{-8} \times 40 \times 10^3 = 4 \times 10^{-4} \text{ Watts}$$

If the noise limiting filter has bandwidth B, then the pre-detection noise power is

$$P_n = 2\int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} \frac{N_0}{2} df = N_0 B = 2 \times 10^{-10} B \text{ Watts}$$

In the case of DSB or conventional AM modulation, $B = 2W = 2 \times 10^4$ Hz, whereas in SSB modulation $B = W = 10^4$. Thus, the pre-detection signal to noise ratio in DSB and conventional AM is

$$\text{SNR}_{\text{DSB,AM}} = \frac{P_R}{P_n} = \frac{4 \times 10^{-4}}{2 \times 10^{-10} \times 2 \times 10^4} = 10^2$$

and for SSB

$$\text{SNR}_{\text{SSB}} = \frac{4 \times 10^{-4}}{2 \times 10^{-10} \times 10^4} = 2 \times 10^2$$

2) For DSB, the demodulation gain (see Problem 5.7) is 2. Hence,

$$\text{SNR}_{\text{DSB},o} = 2\text{SNR}_{\text{DSB},i} = 2 \times 10^2$$

3) The demodulation gain of a SSB system is 1. Thus,

$$\text{SNR}_{\text{SSB},o} = \text{SNR}_{\text{SSB},i} = 2 \times 10^2$$

4) For conventional AM with $\alpha = 0.8$ and $P_{M_n} = 0.2$, we have

$$\text{SNR}_{\text{AM},o} = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}\text{SNR}_{\text{AM},i} = 0.1135 \times 2 \times 10^2$$

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Problem 6.9

1) For an FM system that utilizes the whole bandwidth $B_c = 2(\beta_f + 1)W$, therefore

$$2(\beta_f + 1) = \frac{100}{4} \Longrightarrow b_f = 11.5$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{FM}} = 3\frac{A_c^2}{2}\left(\frac{\beta_f}{\max[|m(t)|]}\right)^2 \frac{P_M}{N_0 W} = 3\frac{A_c^2}{2}\beta_f^2 \frac{P_{M_n}}{N_0 W}$$

For an AM system

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{A_c^2 \alpha^2 P_{M_n}}{2N_0 W}$$

Hence,

$$\frac{\left(\frac{S}{N}\right)_{o,\text{FM}}}{\left(\frac{S}{N}\right)_{o,\text{AM}}} = \frac{3\beta_f^2}{\alpha^2} = 549.139 \sim 27.3967 \text{ dB}$$

2) Since the PM and FM systems provide the same SNR

$$\left(\frac{S}{N}\right)_{o,\text{PM}} = \frac{k_p^2 A_c^2}{2}\frac{P_M}{N_0 W} = \frac{3k_f^2 A_c^2}{2W^2}\frac{P_M}{N_0 W} = \left(\frac{S}{N}\right)_{o,\text{FM}}$$

or

$$\frac{k_p^2}{3k_f^2} = \frac{1}{W^2} \Longrightarrow \frac{\beta_p^2}{3\beta_f^2 W^2} = \frac{1}{W^2}$$

Hence,

$$\frac{\text{BW}_{\text{PM}}}{\text{BW}_{\text{FM}}} = \frac{2(\beta_p + 1)W}{2(\beta_f + 1)W} = \frac{\sqrt{3}\beta_f + 1}{\beta_f + 1}$$

Problem 6.10

1) The received signal power can be found from

$$10 \log \frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8} P_T = 10^{-4} \text{ Watts}$$

$$\left(\frac{S}{N}\right)_o = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}\left(\frac{S}{N}\right)_b = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}\frac{P_R}{N_0 W}$$

Thus, with $P_R = 10^{-4}$, $P_{M_n} = 0.1$, $\alpha = 0.8$ and

$$N_0 W = 2 \times 0.5 \times 10^{-12} \times 5 \times 10^3 = 5 \times 10^{-9}$$

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we find that

$$\left(\frac{S}{N}\right)_o = 1204 \quad 30.806 \text{ db}$$

2) Using Carson's rule, we obtain

$$B_c = 2(\beta+1)W \Longrightarrow 100 \times 10^3 = 2(\beta+1)5 \times 10^3 \Longrightarrow \beta = 9$$

We check now if the threshold imposes any restrictions.

$$\left(\frac{S}{N}\right)_{b,th} = \frac{P_R}{N_0 W} = 20(\beta+1) = \frac{10^{-4}}{10^{-12} \times 5 \times 10^3} \Longrightarrow \beta = 999$$

Since we are limited in bandwidth we choose $\beta = 9$. The output signal to noise ratio is

$$\left(\frac{S}{N}\right)_o = 3\beta^2 0.1 \left(\frac{S}{N}\right)_b = 3 \times 9^2 \times 0.1 \times \frac{10^5}{5} = 486000 \quad 56.866 \text{ db}$$

Problem 6.11

1) First we check whether the threshold or the bandwidth impose a restrictive bound on the modulation index. By Carson's rule

$$B_c = 2(\beta+1)W \Longrightarrow 60 \times 10^3 = 2(\beta+1) \times 8 \times 10^3 \Longrightarrow \beta = 2.75$$

Using the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta+1)P_{M_n}$$

with $\left(\frac{S}{N}\right)_o = 10^4$ and $P_{M_n} = \frac{1}{2}$ we find

$$10^4 = 30\beta^2(\beta+1) \Longrightarrow \beta = 6.6158$$

Since we are limited in bandwidth we choose $\beta = 2.75$. Then,

$$\left(\frac{S}{N}\right)_o = 3\beta^2 P_{M_n} \left(\frac{S}{N}\right)_b \Longrightarrow \left(\frac{S}{N}\right)_b = \frac{2 \times 10^4}{3 \times 2.75^2} = 881.542$$

Thus,

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = 881.542 \Longrightarrow P_R = 881.542 \times 2 \times 10^{-12} \times 8 \times 10^3 = 1.41 \times 10^{-5}$$

Since the channel attenuation is 40 db, we find

$$P_T = 10^4 P_R = 0.141 \text{ Watts}$$

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2) If the minimum required SNR is increased to 60 db, then the β from Carson's rule remains the same, whereas from the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta+1)P_{M_n} = 10^6$$

we find $\beta = 31.8531$. As in part 1) we choose $\beta = 2.75$, and therefore

$$\left(\frac{S}{N}\right)_b = \frac{1}{3\beta^2 P_{M_n}}\left(\frac{S}{N}\right)_o = 8.8154 \times 10^4$$

Thus,

$$P_R = N_0 W 8.8154 \times 10^4 = 2 \times 10^{-12} \times 8 \times 10^3 \times 8.8154 \times 10^4 = 0.0014$$

and

$$P_T = 10^4 P_R = 14 \text{ Watts}$$

3) The frequency response of the receiver (de-emphasis) filter is given by

$$H_d(f) = \frac{1}{1 + j\frac{f}{f_0}}$$

with $f_0 = \frac{1}{2\pi \times 75 \times 10^{-6}} = 2100$ Hz. In this case,

$$\left(\frac{S}{N}\right)_{o,\text{PD}} = \frac{(\frac{W}{f_0})^3}{3\left(\frac{W}{f_0} - \arctan\frac{W}{f_0}\right)}\left(\frac{S}{N}\right)_o = 10^6$$

From this relation we find

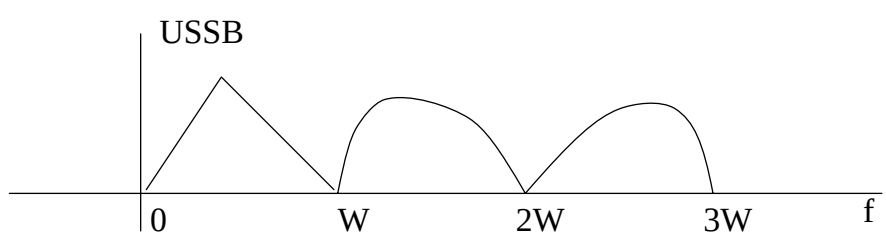
$$\left(\frac{S}{N}\right)_o = 1.3541 \times 10^5 \Longrightarrow P_R = 9.55 \times 10^{-5}$$

and therefore,

$$P_T = 10^4 P_R = 0.955 \text{ Watts}$$

Problem 6.12

1) In the next figure we plot a typical USSB spectrum for $K = 3$. Note that only the positive frequency axis is shown.



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2) The bandwidth of the signal $m(t)$ is $W_m = KW$.

3) The noise power at the output of the LPF of the FM demodulator is

$$P_{n,o} = \int_{-W_m}^{W_m} \mathcal{S}_{n,o}(f)df = \frac{2N_0 W_m^3}{3A_c^2} = \frac{2N_0 W^3}{3A_c^2} K^3$$

where A_c is the amplitude of the FM signal. As it is observed the power of the noise that enters the USSB demodulators is proportional to the cube of the number of multiplexed signals.

The ith message USSB signal occupies the frequency band $[(i-1)W, iW]$. Since the power spectral density of the noise at the output of the FM demodulator is $\mathcal{S}_{n,o}(f) = \frac{N_0}{A_c^2} f^2$, we conclude that the noise power at the output of the ith USSB demodulator is

$$P_{n,o_i} = \frac{1}{4} P_{n_i} = \frac{1}{4} 2 \int_{-(i-1)W}^{iW} \frac{N_0}{A_c^2} f^2 df = \frac{N_0}{2A_c^2} \frac{1}{3} f^3 \Big|_{-(i-1)W}^{iW} = \frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)$$

Hence, the noise power at the output of the ith USSB demodulator depends on i.

4) Using the results of the previous part, we obtain

$$\frac{P_{n,o_i}}{P_{n,o_j}} = \frac{3i^2 - 3i + 1}{3j^2 - 3j + 1}$$

5) The output signal power of the ith USSB demodulator is $P_{s_i} = \frac{A_i^2}{4} P_{M_i}$. Hence, the SNR at the output of the ith demodulator is

$$\text{SNR}_i = \frac{\frac{A_i^2}{4} P_{M_i}}{\frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)}$$

Assuming that P_{M_i} is the same for all i, then in order to guarantee a constant SNR_i we have to select A_i^2 proportional to $3i^2 - 3i + 1$.

Problem 6.13

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

Since the real part of the roots is negative, the poles lie in the left half plane and therefore, the system is stable.

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2) Writing the denominator in the form

$$D = s^2 + 2\zeta\omega_n s + \omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\zeta = \frac{1}{\sqrt{2}}$.

Problem 6.14

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{K}{\tau_1 s^2 + s + K} = \frac{\frac{K}{\tau_1}}{s^2 + \frac{1}{\tau_1}s + \frac{K}{\tau_1}}$$

The gain of the system at $f = 0$ is

$$|H(0)| = |H(s)|_{s=0} = 1$$

2) The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-1 \pm \sqrt{1 - 4K\tau_1}}{2\tau_1} =$$

In order for the system to be stable the real part of the poles must be negative. Since K is greater than zero, the latter implies that τ_1 is positive. If in addition we require that the damping factor $\zeta = \frac{1}{2\sqrt{\tau_1 K}}$ is less than 1, then the gain K should satisfy the condition

$$K > \frac{1}{4\tau_1}$$

Problem 6.15

The transfer function of the RC circuit is

$$G(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

From the last equality we identify the time constants as

$$\tau_2 = R_2 C, \qquad \tau_1 = (R_1 + R_2)C$$

Problem 6.16

Assuming that the input resistance of the operational amplifier is high so that no current flows through it, then the voltage-current equations of the circuit are

$$\begin{aligned} V_2 &= -AV_1 \\ V_1 - V_2 &= \left(R_1 + \frac{1}{Cs}\right) i \\ V_1 - V_0 &= iR \end{aligned}$$

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where, V_1, V_2 is the input and output voltage of the amplifier respectively, and V_0 is the signal at the input of the filter. Eliminating i and V_1, we obtain

$$\frac{V_2}{V_1} = \frac{\frac{R_1+\frac{1}{Cs}}{R}}{1+\frac{1}{A}-\frac{R_1+\frac{1}{Cs}}{AR}}$$

If we let $A \to \infty$ (ideal amplifier), then

$$\frac{V_2}{V_1} = \frac{1+R_1Cs}{RCs} = \frac{1+\tau_2 s}{\tau_1 s}$$

Hence, the constants τ_1, τ_2 of the active filter are given by

$$\tau_1 = RC, \qquad \tau_2 = R_1C$$

Problem 6.17

1) The power is given by

$$P = \frac{V^2}{R}$$

Hence, with $R = 50$, $P = 20$, we obtain

$$V^2 = PR = 20 \times 50 = 1000 \Longrightarrow V = 1000^{\frac{1}{2}} = 31.6228 \text{ Volts}$$

2) The current through the load resistance is

$$i = \frac{V}{R} = \frac{31.6228}{50} = 0.6325 \text{ Amp}$$

3) The dBm unit is defined as

$$\text{dBm} = 10\log\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log(\text{actual power in Watts})$$

Hence,

$$P = 30 + 10\log(50) = 46.9897 \text{ dBm}$$

Problem 6.18

1) The overall loss in 200 Km is $200 \times 20 = 400$ dB. Since the line is loaded with the characteristic impedance, the delivered power to the line is twice the power delivered to the load in absence of line loss. Hence, the required power is $20 + 400 = 420$ dBm.

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2) Each repeater provides a gain of 20 dB, therefore the spacing between two adjacent receivers can be up to $20/2 = 10$ Km to attain a constant signal level at the input of all repeaters. This means that a total of $200/10 = 20$ repeaters are required.

Problem 6.19

1) Since the noise figure is 2 dB, we have

$$10 \log \left(1 + \frac{\mathcal{T}_e}{290}\right) = 2$$

and therefore $\mathcal{T}_e = 169.62°$ K.

2) To determine the output power we have

$$P_{no} = \mathcal{G} k B_{\text{neq}} (\mathcal{T} + \mathcal{T}_e)$$

where $10 \log \mathcal{G} = 35$, and therefore, $\mathcal{G} = 10^3.5 = 3162$. From this we obtain

$$P_{no} = 3162 \times 1.38 \times 10^{-23} \times 10 \times 10^6 (169.62 + 50) = 9.58 \times 10^{-11} \quad \text{Watts} \sim -161.6 \quad \text{dBm}$$

Problem 6.20

Using the relation $P_{n_o} = \mathcal{G} k B_{neq} (\mathcal{T} + \mathcal{T}_e)$ with $P_{n_o} = 10^8 k T_0$, $B_{neq} = 25 \times 10^3$, $\mathcal{G} = 10^3$ and $\mathcal{T} = T_0$, we obtain

$$(10^8 - 25 \times 10^6) T_0 = 25 \times 10^6 \mathcal{T}_e \Longrightarrow \mathcal{T}_e = 3T_0$$

The noise figure of the amplifier is

$$F = \left(1 + \frac{\mathcal{T}_e}{\mathcal{T}}\right) = 1 + 3 = 4$$

Problem 6.21

The proof is by induction on m, the number of the amplifiers. We assume that the physical temperature $\mathcal{T}$ is the same for all the amplifiers. For $m = 2$, the overall gain of the cascade of the two amplifiers is $\mathcal{G} = \mathcal{G}_1 \mathcal{G}_2$, whereas the total noise at the output of the second amplifier is due to the source noise amplified by two stages, the first stage noise excess noise amplified by the second stage, and the second stage excess noise. Hence,

$$\begin{aligned} P_{n_2} &= \mathcal{G}_1 \mathcal{G}_2 P_{n_s} + \mathcal{G}_2 P_{n_i,1} + P_{n_i,2} \\ &= \mathcal{G}_1 \mathcal{G}_2 k \mathcal{T} B_{neq} + \mathcal{G}_2 (\mathcal{G}_1 k B_{neq} \mathcal{T}_{e_1}) + \mathcal{G}_2 k B_{neq} \mathcal{T}_{e_2} \end{aligned}$$

The noise of a single stage model with effective noise temperature $\mathcal{T}_e$, and gain $\mathcal{G}_1 \mathcal{G}_2$ is

$$P_n = \mathcal{G}_1 \mathcal{G}_2 k B_{neq} (\mathcal{T} + \mathcal{T}_e)$$

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Equating the two expressions for the output noise we obtain

$$\mathcal{G}_1\mathcal{G}_2(\mathcal{T}+\mathcal{T}_e)=\mathcal{G}_1\mathcal{G}_2\mathcal{T}+\mathcal{G}_1\mathcal{G}_2\mathcal{T}_{e_1}+\mathcal{G}_2\mathcal{T}_{e_2}$$

or

$$\mathcal{T}_e=\mathcal{T}_{e_1}+\frac{\mathcal{T}_{e_2}}{\mathcal{G}_1}$$

Assume now that if the number of the amplifiers is $m-1$, then

$$\mathcal{T}_e'=\mathcal{T}_{e_1}+\frac{\mathcal{T}_{e_2}}{\mathcal{G}_1}+\cdots\frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1\cdots\mathcal{G}_{m-2}}$$

Then for the cascade of m amplifiers

$$\mathcal{T}_e=\mathcal{T}_e'+\frac{\mathcal{T}_{e_m}}{\mathcal{G}'}$$

where $\mathcal{G}'=\mathcal{G}_1\cdots\mathcal{G}_{m-1}$ is the gain of the $m-1$ amplifiers and we have used the results for $m=2$. Thus,

$$\mathcal{T}_e=\mathcal{T}_{e_1}+\frac{\mathcal{T}_{e_2}}{\mathcal{G}_1}+\cdots\frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1\cdots\mathcal{G}_{m-2}}+\frac{\mathcal{T}_{e_m}}{\mathcal{G}_1\cdots\mathcal{G}_{m-1}}$$

Proof of Fries formula follows easily with the substitution $F_k=\left(1+\frac{\mathcal{T}_{ek}}{\mathcal{T}_0}\right)$ into the above equation.

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